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Some questions on convergence of multiple Fourier series

SYNOPSIS

of the thesis for the degree of candidate of
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Ատենախոսության թեման հաստատվել է Երևանի Պետական Համալսարանում

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Պաշտոնական ընդդիմախոսներ՝	ֆիզ.-մաթ. գիտ. դոկտոր Ս. Ա. Եպիսկոպոսյան ֆիզ.-մաթ. գիտ. թեկնածու Ս. Ա. Սարգսյան
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The topic of the thesis was approved in Yerevan State University

Scientific advisor	Doctor of phys.-math. sciences M. G. Grigoryan
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The defense will be held on April 21, 2020 at 15^{00} at a meeting of the specialized council of mathematics 050, operating at the Yerevan State University (0025, 1 Alek Manukyan St, Yerevan).

The thesis can be found at the YSU library.

The synopsis was sent on March 12, 2020.

Scientific secretary
of specialized council T. N. Harutyunyan

Overview

Relevance of the topic. The problems of the existence of so called 'universal functions' and the 'universal series' are classical, and there is an extensive literature on the theory of functions which are universal in different senses. Many results on the existence of the universal functions are obtained by famous mathematicians such as Birkhoff, MacLane, Marcinkiewicz, Grosse-Erdmann and Luh. Important results on the existence of universal series by different orthonormal systems are obtained by the prominent mathematicians Menshov, Talalyan, Ulyanov and their students. Last years, Grigoryan and his students-coauthors constructed functions whose universality was introduced through Fourier series with respect to the classical systems (see [8]–[14]).

Goals.

1. Construct an integrable function of two variables which double Fourier-Walsh series converges both by rectangles and by spheres, the coefficients on the spectrum are positive and arranged in decreasing order over all directions, and after the choice of proper signes for its Fourier coefficients the spherical partial sums of the newly recieved series are dense in $L^p[0,1]^2$, when $p \in (0,1)$.
2. Construct a weighted space and a series in that space by the double Walsh system, which is universal with respect to the subseries, which coefficients are decreasing over all directions and belong to $l_q,\ q > 2$.
3. Study the behavior of the Fourier coefficients with respect to the double Vilenkin system, as well as almost everywhere convergence of the spherical partial sums of the double Fourier-Vilenkin series after modification of functions.
4. Study the convergence by measure of the Fourier integral spherical means of Riesz at a critical exponent $\delta = 1/2$ after changing the values of the integrable function on the given set of a small measure.

Research methods. Methods of theory of functions and real analysis.

Scientific novelty. All results are new.

Theoretical and practical value. All the results and developed methods represent theoretical interest.

Approbation of results. Most of the results were reported in the following conferences: International Conference "Harmonic analysis and approximations", VII, 2018 (Tsaghkadzor, Armenia), International Conference dedicated to the 100th anniversary of Yerevan State University (Yerevan, Armenia), in seminars of the chair of Higher Mathematics of faculty of Physics.

Publications. The main results of this thesis are published in 6 works (4 papers and 2 conference abstracts), which are listed at the end of references.

Structure and volume of the thesis. The thesis consists of introduction, four chapters, conclusion and bibliography with 67 items. Total number of pages is 71.

The content of the work

The thesis is devoted to the constructing of the functions which are universal in some sense with respect to the double Walsh system, and to the convergence of the Fourier series by the double Walsh and Vilenkin systems. Note that the problems of the existence of so called ‘universal functions’ and the ‘universal series’ are classical, and there is an extensive literature on the theory of functions, which are universal in different senses.

The first example is due to Birkhoff [1], who has proved the existence of an entire function $f(z)$ with the property that for an arbitrary entire function $g(z)$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of the natural numbers such that $\{f(z+n_k)\}_{k=1}^{\infty}$ converges to $g(z)$ compactly on $\mathbb{C}$.

Hence the sequence $\{f(z+n)\}_{n=1}^{\infty}$ of ‘additive translates’ is dense in the space of all entire functions endowed with the topology of compact convergence.

Zappa [2] has established a theorem analogous to that of Birkhoff for the punctured plane $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. He has proved the existence of a holomorphic function $f(z)$ on $\mathbb{C}^*$ with the property that for any compact set $K \in \mathbb{C}^*$, whose complement is connected in $\mathbb{C}^*$, the sequence $\{f(nz)\}_{n=1}^{\infty}$ is dense in $A(K)$ ($A(K)-$ the space of all continuous functions $f(z)$ on K and holomorphic in K).

In [3] MacLane has proved a similar result for another type of universality, namely, there exists an entire function $f(z)$ which is universal with respect to derivatives, that is, for every entire function $g(z)$ and for each number $r > 0$, there exists an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ so that the sequence $\{f^{(n_k)}(z)\}_{k=1}^{\infty}$ converges to $g(z)$ uniformly on $|z| \leq r$.

In [4] Marcinkiewicz has proved that for any nonzero sequence $h_n \to 0$ there exists a continuous function $F \in C[0,1]$, $F:[0,1] \to \mathbb{R}$ having the property: for any measurable function $f(x)$: $[0,1]$ there is a subsequence $n_k \nearrow \infty$ such that almost everywhere on $[0,1]$

$$\frac{F(x+h_{n_k}) - F(x)}{h_{n_k}} \to f(x),\ k \to \infty.$$

This continuous function F is called a universal primitive function with respect to the given sequence $\{h_n\}_{n=1}^{\infty}$ (see also [5]).

In [6] Grosse-Erdmann has proved the existence of an infinitely differentiable function with universal Taylor expansion. Namely, there exists a function $g(x) \in$

$C^\infty(\mathbb{R})$ with $g(0) = 0$ such that the Taylor series at $x_0 = 0$ is locally uniformly universal in $C(\mathbb{R})$, that is, for any function $f(x) \in C(\mathbb{R})$ with $f(0) = 0$ and for a number $r > 0$ there exists a subsequence

$$S_{n_k}(g, 0) = \sum_{m=1}^{n_k} \frac{g^{(m)}(0)}{m!} x^m$$

of partial sums of the Taylor series for $g(x)$ which converges to $f(x)$ uniformly on the interval $(-r, r)$.

In [7] Luh has proved a theorem on the universality of power series $\sum_{k=0}^{\infty} c_k z^k$. Namely, for $r \geq 0$ there exists a power series $\sum_{k=0}^{\infty} c_k z^k$ of radius of convergence r such that for every compact set K in $\{z \in \mathbb{C} : |z| > r\}$ with connected complement and every function $h(z)$ that is continuous on K and holomorphic on the interior of K there exists an increasing sequence $\{N_m\}_{m=1}^{\infty} \in \{0, 1, 2, ...\}$ such that

$$\sum_{k=0}^{N_m} c_k z^k \to h(z)$$

as $m \to \infty$ uniformly on K.

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [0, 1)$ (or $E \subseteq [0, 1) \times [0, 1) = [0, 1)^2$).

Let $L^p[0, 1]$ $(p > 0)$ be the class of the functions $f(x)$ which are measurable on $[0, 1]$ and for which $\int_0^1 |f(x)|^p dx < +\infty$, $\{\varphi_k(x)\}_{k=0}^{\infty}$, the Walsh system, $c_k(g) = \int_0^1 g(x)\varphi_k(x)dx$ are Fourier-Walsh coefficients of the function $g \in L^1[0, 1]$ and $S_n(x, g) = \sum_{k=0}^{n} c_k(g)\varphi_k(x)$.

Let $T = [0, 1]^2$, and let $f(x, y) \in L^p(T)$, $p \in [1, \infty)$, that is,

$$\iint_T |f(x, y)|^p dxdy < \infty.$$

The Fourier coefficients of the function $f \in L^p(T)$, $p \in [1, \infty)$, by the double Walsh system $\{\varphi_k(x)\varphi_s(y)\}_{k,s=0}^{\infty}$ we denote by

$$c_{k,s}(f) = \iint_T f(t, \tau)\varphi_k(t)\varphi_s(\tau)dt\, d\tau.$$

Define the spectrum of f by

$$spec(f) = \{(k, s) : \; c_{k,s}(f) \neq 0, \; k, \; s \in \mathbb{N} \cap \{0\}\}.$$

Definition 1.1. Say that the double Fourier-Walsh series of the function $f \in L^1[0,1)^2$ converges in $L^p[0,1)^2$, $p > 0$, by rectangles if

$$\lim_{N\to\infty, M\to\infty} \iint_T |S_{N,M}(x,y,f) - f(x,y)|^p \, dxdy = 0,$$

and by spheres if

$$\lim_{R\to\infty} \iint_T |S_R(x,y,f) - f(x,y)|^p \, dxdy = 0.$$

Definition 2.1. The nonzero members of a double sequence $\{b_{k,s}\}_{k,s=0}^{\infty}$ are said to be in a monotonically decreasing order over all rays if $b_{k_2,s_2} < b_{k_1,s_1}$, when $k_2 \geq k_1$, $s_2 \geq s_1$, $k_2 + s_2 > k_1 + s_1$ $(b_{k_i,s_i} \neq 0, \ i = 1,2)$.

There were constructed functions whose universality was introduced through Fourier series with respect to the classical systems (see [8–14]). In [8] Grigoryan and Sargsyan have proved the existence of a function $U \in L^1[0,1]$ with Fourier series with respect to the Walsh system being universal in $L^p[0,1]$, $p \in (0,1)$, in sense of signs, that is for each function $f \in L^p[0,1]$, $p \in (0,1)$, one can find numbers $\delta_k = \pm 1$ such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U)\varphi_k(x)$ converges to f in $L^p[0,1]$: i.e.

$$\lim_{m\to\infty} \int_0^1 \left| \sum_{k=0}^{m} \delta_k c_k(U)\varphi_k(x) - f(x) \right|^p dx = 0,$$

where $c_k(U) = \int_0^1 U(x)\varphi_k(x)dx$, $k \geq 0$, are the Fourier -Walsh coefficients of the function $U(x)$.

In [14] the following result is obtained:
for each number $\varepsilon > 0$ there exists a measurable set $E_\varepsilon \subset [0,1)^2$ with measure $|E_\varepsilon| > 1 - \varepsilon$, such that for each function $f \in L^1[0,1)^2$ one can find a function $\tilde{f} \in L^1[0,1)^2$, coinciding with $f(x,y)$ on E_ε, such that

1) rectangular and spherical partial sums of the Fourier series of $\tilde{f}$ by the double Walsh system converge to it by $L^1[0,1)^2$ norm;

2) its nonzero Fourier coefficients stand in strictly decreasing order over all rays by the absolute values;

3) it is universal (by rectangles and by spheres) for each class $L^p[0,1)^2$ with respect to the double Walsh system in the sense of signs of Fourier coefficients that is, for any $f \in L^p[0,1)^2$ one can find signs $\{\delta_{k,s} = \pm 1\}_{k,s=1}^{\infty}$ such that the series $\sum_{k,s=0}^{\infty} c_{k,s}(\tilde{f})\varphi_k(x)\varphi_s(y)$ converges to f both by rectangles and by spheres.

It is well known that the Fourier series of each integrable function by the Walsh system and by the trigonometric system converges in $L^p[0,1)$ for every $p \in (0,1)$. From this result it follows that there doesn't exist a function which Fourier series partial sums by the Walsh system (and also by the trigonometric system) are dense in $L^p[0,1)$ for some $p \in (0,1)$.

The following question arises naturally, which is an open problem:

Question 1. Does there exist a bounded orthonormal system $\{\varphi_k(x)\}_{k=0}^{\infty}$ such that a function $U \in L^1[0,1]$ can be constructed the partial sums of the Fourier series of which by the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ would be dense in $L^p[0,1]$ for some $p \in (0,1)$?

Note that Grigoryan (see [9], [12]) has constructed a function $U \in L^1[0,1]$ such that after a choice of the corresponding signs for its Fourier coefficients, the partial sums of the obtained series $\sum_{k=0}^{\infty} \delta_k c_k(U)\varphi_k(x)$ ($\delta_k = \pm 1$) are dense in $L^p[0,1]$, for $p \in (0,1)$.

In the first chapter of the thesis we consider the following question: whether it is possible to get the similar result in two-dimensional case. In the same chapter a universal function $U(x,y) \in L^1[0,1)^2$ is constructed and a sequence of signs $\{\delta_{k,s};\ \delta_{k,s} = \pm \mathit{1}\}_{k=0}^{\infty}$ is chosen, so that the spherical partial sums of the series $\sum_{k,s=0}^{\infty} \delta_{k,s} c_{k,s}(U)\varphi_k(x)\varphi_s(y)-$ are dense in $L^p[0,1)^2$ for each $p \in (0,1)$ (here $c_{k,s}(U)$ are the Fourier coefficients of the function $U(x,y)$ by the Walsh double system).

In the first chapter the following theorem is proved:

Theorem 1.1. There exist numbers $\{\delta_{k,s} = \pm 1\}_{k,s=0}^{\infty}$ and a function $U \in L^1[0,1)^2$ ($\mathrm{supp}U \subset ([0,\varepsilon] \times [0,1]) \cup ([0,1] \times ([0,\varepsilon]))$ with $\varepsilon \in (0,1)$ given) whose Fourier series by the double Walsh system converges to itself both by rectangles and spheres. Moreover, the coefficients of the series on the spectrum are positive and arranged in decreasing order over all directions, and the spherical partial sums of the series $\sum \delta_{k,s} c_{k,s}(U)\varphi_k(x)\varphi_s(y)$ are dense in $L^p(T)$, for each $p \in (0,1)$, where $c_{k,s}(U)$ are the Fourier-Walsh coefficients of the function U.

Note that a wide function class possess this property, and the structure of these universal functions can be described from the viewpoint of well-known classical theorems by Luzin [15] and Menshov [16], [25] 'on correction of functions' (see also [34]-[39]).

Theorem (Lusin's C-property). For every measurable, almost everywhere finite function f on $[0,1]$ and for every $\varepsilon > 0$ there exists a measurable set $E \subset [0,1]$

with $|E| > 1-\varepsilon$ and a continuous function $g(x),\ x \in [0,1]$, that coincides with $f(x)$ on E.

Theorem (Menshov's C-strong property). For every measurable, almost everywhere finite function f on $[0,2\pi]$ and for every $\varepsilon > 0$ there is a continuous function f_ε such that $|x \in [0,2\pi];\ f_\varepsilon(x) = f(x)| < \varepsilon$ and the Fourier series of the function f_ε converges uniformly in $[0,2\pi]$.

The main result of this chapter is the theorem below.

Theorem 1.2. For any number $0 < \varepsilon < 1$ there exists a measurable set $E_\varepsilon \subset [0,1)^2$ with the measure $|E_\varepsilon| > 1-\varepsilon$ and such that for any function $f \in L^1(T)$ one can find numbers $\{\delta_{k,s} = \pm 1\}_{k,s=0}^{\infty}$ and a function $\tilde{f} \in L^1(T)$, which coincides with f on E_ε which Fourier series by double Walsh system converges both by rectangles and by spheres, the coefficients on the spectrum are in decreasing order over all directions by the absolute values, and the spherical partial sums of the series $\sum \delta_{k,s} c_{k,s}(\tilde{f}) \varphi_k(x) \varphi_s(y)$ are dense in $L^p(T)$ for each $p \in (0,1)$.

Remark 1.1. It should be noted that the theorem is in some sense the best possible (unimprovable) because for arbitrary number $p \geq 1$ and any bounded orthonormal system $\{\varphi_n(x)\}$, there doesn't exist a function $U \in L^1(T)$ and numbers $\{\delta_{k,s} = \pm 1\}_{k,s=0}^{\infty}$ for which the spherical partial sums of the series $\sum \delta_{k,s} c_{k,s}(U) \varphi_k(x) \varphi_s(y)$ would be dense in $L^p(T)$.

It would be of interest to answer the following question:

Question 1.1. Does there exist a function $U \in L^1(T)$ for which the rectangular or spherical partial sums of Fourier series by the double Walsh system $\{\varphi_k(x)\varphi_s(y)\}_{k,s=0}^{\infty}$ or by the trigonometric system $\{e^{2\pi kix} e^{2\pi siy}\}_{k,s=0}^{\infty}$ would be dense in $L^p(T)$, for some $p \in (0,1)$?

Note that a number of classical results (such as the L. Carleson theorem [17]: Fourier series of any function $f \in L^2[0,2\pi]$ by the trigonometric system converges almost everywhere on $[0,2\pi]$; the M. Riesz theorem [18]: Fourier series of any function $f \in L^p[0,2\pi],\ p > 1$, by the trigonometric system converges by $L^p[0,2\pi]$ norm; the A. M. Kolmogorov theorem [19]: Fourier series of each function $f \in L^1[0,2\pi]$ by the trigonometric system converges in $L^p[0,2\pi],\ p \in (0,1)$, metric) cannot be transfered from the one-dimensional case to the two-dimensional one. In this respect, even different (spherical, rectangular, square, etc.) partial sums differ sharply from each other in their properties in matters like convergence in $L^p[0,2\pi],\ p \geq 1$, and convergence almost everywhere.

For example, in [20] Fefferman has obtained the following results:

1) For each $p \neq 2$ there exists a function $f_p(x, y)$ from $L^p(0, 2\pi)^2$, for which spherical partial sums of Fourier series by the trigonometric system do not converge in $L^p(0, 1)^2$ norm.

2) There exists a continuous function $f(x, y)$, for which rectangular partial sums of the double Fourier series by the trigonometric system diverge at each point in $(0, 2\pi)^2$.

In [23] M. Grigoryan has proved the existence of such a function $f_0 \in L^1(0, 2\pi)^2$ that the spherical partial sums of its double Fourier series by the trigonometric system diverges in $L^p(0, 2\pi)^2$ metrics for any $p \in (0, 1)$.

In [22] Oniani has constructed a function $f \in L^1[0, 1)^2$ such that the rectangular and spherical common terms of double Fourier-Haar series of f are unbounded a.e. on $[0, 1)^2$.

In the work [21] Getsadze has proved that for any uniformly bounded complete in $L^2[0, 1]$ orthonormal system $\{\psi_k(x)\}_{k=1}^{\infty}$ there exists such a function $f(x, y)$ from $L^1(0, 1)^2$ that the double Fourier series of which by the system $\{\psi_k(x)\psi_n(y)\}_{k=1}^{\infty}$ diverges in measure by squares.

Note that the behavior of spherical partial sums of double Fourier and Fourier – Walsh series of continuous functions is not known yet.

Note also that in one-dimensional case the Fourier series of each $f \in L^1[0, 1)$ with respect to the Haar system converges a.e..

In [24] Harris has constructed a function $f \in L^p[0, 1)^2$, for any $1 \leq p < 2$, such that the Fourier-Walsh series of $f(x, y)$ in the Walsh double system diverges almost everywhere and in the $L^p[0, 1)^2$-norm with respect to spheres. From this result it follows that it is impossible to find a double series in the double Walsh system for every function $f(x, y) \in L^p[0, 1)^2$ that converges to the function $f(x, y)$ in the $L^p[0, 1)^2$-norm or almost everywhere with respect to spheres.

In the thesis we prove that for any $\varepsilon > 0$ there exists a measurable set $E \subset [0, 1)^2$ with measure $|E| > 1 - \varepsilon$, such that for any function $f(x, y) \in L^p(E)$, $p \geq 1$, one can find a series $\sum_{k,n=0}^{\infty} b_{k,n}\varphi_k(x)\varphi_n(y)$ with respect to the Walsh double system which converges to the function $f(x, y)$ in the $L^p(E)$-norm with respect to spheres, that is,

$$\lim_{R\to\infty}\int\int_E\left|\sum_{0\le k^2+s^2\le R^2} b_{k,s}\varphi_k(x)\varphi_s(y)-f(x,y)\right|^p dxdy=0.$$

Moreover, in the second chapter of the thesis we prove Theorem 2.1, and this result follows from the stronger Theorem 2.2 which is also proved in the same chapter. In the second chapter we construct a weighted space $L_p^{\mu}[0,1)^2,\ p\ge 1$, and a series $\Sigma_{k,s=0}^{\infty} b_{k,s}\varphi_k(x)\varphi_s(y)$ by the double Walsh system, which is universal with respect to the subseries, the coefficients are decreasing over all directions and belong to $l_q,\ q>2$. Note that many authors have got important results on the universality of series both by the classical and general orthonormal systems (see [26-39]).

Let $\mu(x,y)$ be a positive Lebesgue-measurable function (weight function) defined on $[0,1)^2$. By $L_\mu^p[0,1)^2$ we denote the space of all measurable functions on $[0,1)^2$ with the norm

$$\|.\|_{L_\mu^p}=\left(\int_0^1\int_0^1 |.|^p\ \mu(x,y)dxdy\right)^{\frac{1}{p}}<\infty:\ p\in[1,\infty).$$

In the sequel, we will accept the terms "measure" and "measurable" in the Lebesgue sense.

Let $\Phi=\{\varphi_k(x)\}_{k=1}^{\infty}$ be the Walsh system. This system forms a basis in the spaces $L^p[0,1)$ for all $p>1$, that is, any function $f(x)\in L^p[0,1)$ can be uniquely represented by the series $\sum_{k=0}^{\infty} c_k(f)\varphi_k(x)$ which converges to f in the $L^p[0,1)$-norm, where $c_k(f)=\int_0^1 f(x)\varphi_k(x)dx$. One could think of $\{c_k(f)\}_{k=0}^{\infty}$ as the sequence of Fourier coefficients of f with respect to the Walsh system.

Definition 2.2. We will say that the Fourier-Walsh double series of a function $f(x,y)\in L_\mu^p[0,1)^2\cap L^1[0,1)^2$ converges to the function $f(x,y)$ in the $L_\mu^p[0,1)^2$-norm with respect to spheres if

$$\lim_{R\to\infty}\left(\int_0^1\int_0^1 |S_R(x,y,f)-f(x,y)|^p\ \mu(x,y)dxdy\right)^{\frac{1}{p}}=0.$$

The definition with respect to rectangles will be given in the same way.

Theorem 2.1. For any $\varepsilon>0$ there exist a measurable set $E\subset[0,1)^2$ with measure $|E|>1-\varepsilon$ and a weight function $\mu(x,y)$ with $|\{(x,y)\in[0,1)^2;\ \mu(x,y)=1\}|>1-\varepsilon$, $0<\mu(x,y)\le 1,\ (x,y)\in[0,1)^2$, such that for every function $f(x,y)\in$

$L^p_\mu[0,1)^2$ one can find a function $g(x,y) \in L^p_\mu[0,1)^2 \cap L^1[0,1)^2$, coinciding with $f(x,y)$ on E, whose Fourier series $\sum_{k,s=0}^{\infty} c_{k,s}(g)\varphi_k(x)\varphi_s(y)$ in the Walsh double system $\{\varphi_k(x)\varphi_s(y)\}_{k,s=0}^{\infty}$ converges to the function $g(x,y)$ in the $L^p_\mu[0,1)^2$-norm with respect to spheres and the non-zero coefficients in $\{|c_{k,s}(g)|\}_{k,s=0}^{\infty}$ are in decreasing order over all rays.

Theorem 2.2. For any $0 < \varepsilon < 1$ and each function $f \in L^\infty[0,1]$ one can find a function $g \in L^\infty[0,1)$, $|\{x \in [0,1) \quad g \neq f\}| < \varepsilon$, such that the sequence $\{|c_k(g)|\ ,\ k \in spec(g)\}$ is monotonically decreasing.

Theorem 2.3. For any $\varepsilon > 0$ there exist a measurable set $E \subset [0,1)$ with measure $|E| > 1-\varepsilon$ and a weight function $\mu(x)$ with $|\{x \in [0,1);\ \mu(x) = 1\}| > 1-\varepsilon$, $0 < \mu(x) \leq 1,\ x \in [0,1)$, such that for every function $f(x) \in L^p_\mu[0,1)$ one can find a function $g(x) \in L^p_\mu[0,1) \cap L^1[0,1)$ coinciding with $f(x)$ on E whose Fourier series $\sum_{k=0}^{\infty} c_k(g)\varphi_k(x)$ in the Walsh system $\{\varphi_k(x)\}_{k=0}^{\infty}$ converges to the function $g(x)$ in the $L^p_\mu[0,1)$-norm and the non-zero coefficients in $\{|c_k(g)|\}_{k=0}^{\infty}$ are in decreasing order.

Theorem 2.4. For any $\varepsilon > 0$ there exist a measurable set $G \subset [0,1)$ with measure $|G| > 1-\varepsilon$, such that for each function $f \in L^1[0,1]$ there exists a function $g \in L^1[0,1]$ equal to $f(x)$ on G and with Fourier series with respect to the Walsh system convergent almost everywhere.

Theorem 2.5. Let $\Phi = \{\varphi_k(x)\}$ be the Walsh system and let $\varepsilon \in (0,1)$. Then, there exist

· a double sequence $\{b_{k,s}\}_{k,s=0}^{\infty}$ with $\sum_{k,s=1}^{\infty} |b_{k,s}|^r < \infty$, for all $r > 2$ and non-zero members in $\{|b_{k,s}|\}_{k,s=0}^{\infty}$, which are in decreasing order over all rays,

· a measurable set $G \subset [0,1)^2$ with measure $|G| > 1-\varepsilon$,

· a measurable function $\mu(x,y)$ with $|\{(x,y) \in [0,1)^2;\ \mu(x,y) = 1\}| > 1-\varepsilon,\ 0 < \mu(x,y) \leq 1,\ (x,y) \in [0,1)^2$, with the following property: for each $p \in [1,\infty)$ and for every function $f(x,y) \in L^p_\mu[0,1)^2$ one can find a function $g(x,y) \in L^p_\mu[0,1)^2 \cap L^1[0,1)^2$ coinciding with $f(x,y)$ on G, whose Fourier series $\sum_{k,s=0}^{\infty} c_{k,s}(g)\varphi_k(x)\varphi_s(y)$ in the Walsh double system $\{\varphi_k(x)\varphi_s(y)\}_{k,s=0}^{\infty}$ converges to the function $g(x,y)$ both in the $L^p_\mu[0,1)^2$-norm and $L^1[0,1)^2$-norm with respect to spheres and $c_{k,s}(g) = b_{k,s},\ for\ all\ (k,s) \in spec(g)$.

Question 2.1. Which one of the above formulated theorems 1-5 does hold for the trigonometric system?

The third chapter of the thesis is on the behavior of the Fourier coefficients with respect to the double Vilenkin system, as well as almost everywhere convergence of the spherical partial sums of the double Fourier-Vilenkin series after modification of functions. The theory of such systems have been introduced by Vilenkin in 1946 (see [40], [42]). There are interesting results for Vilenkin system (see [40]-[46]).

In 1957 Watary [44] proved that the bounded Vilenkin system is basis in L^r when $r > 1$. Then, in 1976, Young [45] established the basisity of Vilenkin system in L^r, when $r > 1$, for arbitrary sequence p_k (that is, both for bounded and unbounded Vilenkin systems). Note that the following problem remains open: Is the Fourier series of function from $L^2[0,1)$ with respect to the unbounded Vilenkin systems convergent almost everywhere or not? Note also that in [46] Billard has established that this problem has a positive answer for the Walsh system.

For the bounded type Vilenkin systems it was proved by Gosselin in [41].

Definition 3.1. Given a subset Γ of the natural numbers, its density $\rho(\Gamma)$ is defined as $\rho(\Gamma) = \limsup_{n\to\infty} \frac{\gamma(n)}{n}$, where $\gamma(n)$ is the number of elements in Γ not exceeding n.

The following results are proved here:

Theorem 3.1. Let $\{V_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$, there exists a measurable set $E \subset [0,1)$ of measure $|E| > 1-\varepsilon$ such that for any function $f \in L^1[0,1)$ there exists a function $g \in L^1[0,1)$ such that $f(x) = g(x)$ if $x \in E$, and the elements of the sequence $\{|c_k(g)|, k \in spec(g)\}$ are monotonically decreasing.

Theorem 3.2. Let $\{V_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$, there exist a measurable set $E \subset [0,1)^2$ of measure $|E| > 1-\varepsilon$, and a sequence $R_k \nearrow$ such that for any function $f(x,y) \in L^1(E)$ there exists a function $g(x,y) \in L^1[0,1)^2$ satisfying the following conditions:

1. $g(x,y) = f(x,y)$ on E,
2. the nonzero members of the sequence $\{|c_{k,s}(g)|\}$ are monotonically decreasing in all ways,
3. the subsequences $\{S_{R_k}((x,y),g)\}$ of spherical sums of the function $g(x,y)$ converge to it almost everywhere.

Theorem 3.3. Let $\{V_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$, there exist a measurable set $E \subset [0,1)^2$ of measure

$|E| > 1 - \varepsilon$, and a subset of natural numbers Γ of density 1 such that for any function $f(x,y) \in L^1(E)$ there exists a function $g(x,y) \in L^1[0,1)^2$ satisfying the following conditions:

1. $g(x,y) = f(x,y)$ on E,

2. the nonzero members of the sequence $\{|c_{k,s}(g)|\}$ are monotonically decreasing in all directions,

3. $\lim_{R\in\Gamma,\ R\to\infty} S_R((x,y),g) = g(x,y)$ almost everywhere on $[0,1)^2$, where $S_R((x,y),g) = \sum_{k^2+s^2\le R^2} c_{k,s}(g)V_k(x)V_s(y)$.

Let E^2 is 2-dimentional euclidean space, and

$$L^1\left(E^2\right) \equiv \left\{ f(x,y) : \iint_{E^2} |f(x,y)| dxdy < \infty \right\},$$

where $f(x,y) \in L^1\left(E^2\right)$, and

$$\widehat{f}(t,\tau) = \frac{1}{(2\pi)^2} \iint_{E^2} f(x,y) e^{-i(tx+\tau y)} dxdy$$

is the transformation of Fourier of the function $f(x,y) \in L^1\left(E^2\right)$.

Call the integrals

$$\sigma_R^\delta\left[(x,y),f\right] = \iint_{t^2+\tau^2\le R^2} \left[1 - \frac{t^2+\tau^2}{R^2}\right]^\delta \widehat{f}(t,\tau) e^{i(tx+\tau y)} dt d\tau$$

integral spherical means of Riesz of order δ.

In the fourth chapter, we study the convergence by the measure of the Fourier integral spherical means of Riesz at a critical exponent $\delta = 1/2$ after changing the values of the integrable function on the given set of a small measure.

Many results on the convergence of Fourier integral spherical means and Fourier double series are obtained (see [47–49]).

The following result is proved in this chapter :

Theorem 4.1. Let $f(x,y) \in L^1\left(E^2\right)$ and $Q \subset E^2$ is perfect nowhere dense set. Then we can define a function $F(x,y) \in L^1\left(E^2\right)$ so that $F(x,y) = f(x,y)$ on Q, and its Fourier integral spherical means of Riesz of the order $\delta = \frac{1}{2}$ converge to it by measure.

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Articles

1. M. G. Grigoryan and L. S. Simonyan, *Double universal Fourier series*, Journal of contemporary mathematical analysis, 2019, n. 6, pp. 221-236.

2. M. G. Grigoryan, T. M. Grigoryan, L. S. Simonyan, *Convergence of double Fourier-Walsh series in weighted* $L^p_\mu[0;1)^2$, Springer Proceedings in Mathematics and Statistics, 2, Springer Nature Switzerland AG 2019, pp. 109-137.

3. L. S. Simonyan, *On convergence of double Fourier series with respect to the Vilenkin systems*, Proceedings of the Yerevan State University, 2018, 52(1), pp. 12-18.

4. L. S. Simonyan, *The representation of functions by double Walsh system in weighted spaces* $L^p_\mu[0;1)^2$, Proceedings of the Yerevan State University, 2019, 53(3), pp. 156-162.

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5. L. S. Simonyan, *On convergence of double Fourier series with respect to the Vilenkin systems.* International Conference "Harmonic analysis and approximations", VII, 2018, pp. 71-72.

6. L. S. Simonyan, *On the representation of functions by double Walsh system in weighted* $L^p_\mu[0;1)^2$*-spaces.* International conference dedicated to the 100th anniversary of Yerevan State University, 2019, pp. 52-53.

Ամփոփում

Ատենախոսության մեջ ստացվել են հետևյալ հիմնական արդյունքները.

- Կառուցվել է երկու փոփոխականի ինտեգրելի ֆունկցիա, որի Ֆուրիե-Ոէոլշի կրկնակի շարքը զուգամիտում է ըստ ուղղանկյունների և ըստ սֆերաների, գործակիցները սպեկտրի վրա դասավորված են ըստ բացարձակ արժեքների բոլոր ուղղություններով նվազման կարգով, իսկ նրա Ֆուրիեի գործակիցների համար համապատասխան նշանների ընտրությունից հետո նոր ստացված շարքի սֆերիկ մասնակի գումարները խիտ են $L^p[0,1]^2$-ում, երբ $p \in (0,1)$:

- Կառուցվել է կշռային տարածություն և այդ տարածությունում շարք ըստ Ոէոլշի կրկնակի համակարգի, որն ունիվերսալ է ենթաշարքերի նկատմամբ, որի գործակիցները նվազող են բոլոր ուղղություններով և պատկանում են l_q-ին $q > 2$ դեպքում:

- Նկարագրվել է Վիլենկինի կրկնակի համակարգի նկատմամբ Ֆուրիեի գործակիցների վարքը, ինչպես նաև Ֆուրիե-Վիլենկինի կրկնակի շարքի սֆերիկ մասնակի գումարների համարյա ամենուրեք զուգամիտությունը ֆունկցիաների ձևափոխությունից հետո:

- Նկարագրվել է Ֆուրիեի կրկնակի ինտեգրալների Ռիսսի սֆերիկ միջինների՝ ըստ չափի զուգամիտությունը $\delta = 1/2$ կրիտիկական ցուցիչի դեպքում ինտեգրելի ֆունկցիայի արժեքները տրված փոքր չափի բազմության վրա փոխելուց հետո:

Заключение

В диссертационной работе получены следующие результаты:

- Построена интегрируемая функция, двойной ряд Фурье-Уолша которой сходится как по прямоугольникам, так и по сферам, коэффициенты расположены убывающим порядком по абсолютным значениям по всем направлениям, а после выбора соответствующих знаков для ее коэффициентов Фурье частичные суммы вновь полученного ряда плотны в $L^p[0,1]^2$, при $p \in (0,1)$.

- Построено весовое пространство и ряд в этом пространстве по двойной системе Уолша, который универсален по подсистемам, коэффициенты которого убывают по всем направлениям и принадлежат к l_q при $q > 2$.

- Описано поведение коэффициентов Фурье по двойной системе Виленкина, а также исследована сходимость почти всюду сферических частичных сумм двойного ряда Фурье-Виленкина после преобразования функций.

- Описана сходимость по мере сферических средних Рисса двойных интегралов Фурье при критическом показателе $\delta = 1/2$ после изменениия значений функции на множестве данной малой меры.