ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Ֆելիքս Վարդանի Հայրապետյան

Միավոր շրջանում և կիսահարթությունում հոլոմորֆ, հարմոնիկ և ողորկ ֆունկցիաների որոշ կշռային դասերի կշռային ինտեգրալ ներկայացումներ և հատկություններ

Ա.01.01 – «Մաթեմատիկական անալիզ» մասնագիտությամբ
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YEREVAN STATE UNIVERSITY

Feliks Vardan Hayrapetyan

Weighted integral representations and properties of some weighted classes of holomorphic, harmonic and smooth functions in the unit disc and half-plane

SYNOPSIS

of the thesis for the degree of candidate of
physical and mathematical sciences in the specialty
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Ատենախոսության թեման հաստատվել է ՀՀ ԳԱԱ Մաթեմատիկայի Ինստիտուտում:

Գիտական ղեկավար՝	ֆիզ. մաթ. գիտ. թեկնածու Ա. Հ. Կարապետյան
Պաշտոնական ընդդիմախոսներ՝	ֆիզ. մաթ. գիտ. դոկտոր Ա. Մ. Ջրբաշյան,
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Մասնագիտական խորհրդի գիտական քարտուղար

ֆիզ. մաթ. գիտ. դոկտոր Տ. Ն. Հարությունյան

The topic of the thesis was approved in NAS RA Institute of Mathematics.

Scientific advisor:	candidate of phys.-math. sciences A. H. Karapetyan
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Leading organization:	National Polytechnic University of Armenia

The defense will be held on November 23, 2021 at 15:00 at a meeting of the specialized council of mathematics 050, operating at the Yerevan State University (0025, 1 Alek Manukyan St, Yerevan).

The thesis can be found at the YSU library.

The synopsis was sent on October 14, 2021.

Scientific secretary of specialized council

doctor of phys.-math. sciences T. N. Harutyunyan

Overview

Relevance of the topic. The thesis relates to investigation of growth estimates of classical Blaschke products in the lower half-plane and to weighted integral representations of weighted L^p-classes of holomorphic, harmonic and smooth functions in the unit disc or upper half-plane.

It is well known that due to investigations of G. H. Hardy, J. E. Littlewood, F. Riesz, M. Riesz, L. Bieberbach, A. Zygmund, G. Szegő, I. I. Privalov, R. Nevanlinna, O. Frostman, A. Beurling and others, the famous Blaschke products (1915) played a fundamental role in developing of function theory in the unit disc. In further researches a growth of Blaschke products near the boundary of the unit disc was investigated. More exactly, an integral logarithmic mean $m_q(r, B)$ of order q $(1 \leq q < +\infty)$ is defined to characterize the behaviour of Blaschke product B near the boundary of the unit disc $\mathbb{D}$. The problem of estimating integral logarithmic means of Blaschke product by its zeros counting function was posed by A. Zygmund for $q = 2$. In 1969 this problem was solved by the method of Fourier series by G.R. MacLane and L.A. Rubel [1]. In [2] V.V. Eiko and A.A. Kondratyuk investigated this problem in the general case, when $1 \leq q < +\infty$.

One of the most interesting continuations of the mentioned problems is the consideration of some Hardy type spaces of Blaschke ω-products or Green type ω-potentials, which is done for $q = 2$ in A. M. Jerbashian's and co-authors' recent works [3, 4].

Let us define the quantity index $p(B)$ of Blaschke products for the unit disc as $p(B) = p - 1$, where $p^{-1} + q^{-1} = 1$ and $q = \sup\{s \in [1, \infty) : m_s(r, B) = O(1), r \to 1\}$.

A.A. Kondratyuk and M.O. Girnik [5] based on the asymptotic formulas of R.S. Galoyan [6], constructed Blaschke products of given quantity index for the unit disc. Earlier L.R. Sons [7] had constructed a Blaschke product for which $m_2(r, B) \to \infty$ as $r \to 1$.

The classical Blaschke products for the upper half-plane introduced by R. Nevanlinna (see, for example, [8], [9]) and their generalizations considered by A.M.Jerbashian and G.V. Mikayelyan (see [10] - [12]) are also very important.

In the thesis we use the method of "Fourier transforms for meromorphic functions" developed by G.V. Mikayelyan in [11], [13], [14], to investigate the growth of classical

Blaschke products (rewritten for the lower half-plane) near the boundary of the half-plane and to obtain analogs of the above-mentioned results related to the unit disc.

It is well-known that the Cauchy integral formula (1831) has numerous applications in complex analysis. It makes it possible to obtain the values of function inside a domain by it's integration over the boundary of a domain. Of course, it is supposed that the boundary is piecewise-smooth and the function is holomorphic inside domain and has (in a certain sense) boundary values. First of all we mean the unit disc in the complex plane and the well-known Hardy spaces H^p. Due to fundamental researches of G. H. Hardy, J. E. Littlewood, F. Riesz, M. Riesz, G. Szegő, I. I. Privalov, V. I. Smirnov, R. Nevanlinna the investigation of these spaces opened a new direction in complex analysis. Later on these investigations were significantly developed and generalized. It turns out (unlike the Cauchy formula) that for certain classes of holomorphic functions the values inside of a domain can be obtained by integration of functions over the whole domain. In the investigations of S. Bergman [15] (see also N. Aronszajn [16]) the classes of holomorphic functions square integrable over the given bounded domain were considered. For this classes reproducing kernels were obtained by means of orthonormal systems of holomorphic functions. In the case of the unit disc $\mathbb{D}$ W. Wirtinger [17], G. H. Hardy and J. E. Littlewood [18, 19], M. M. Djrbashian [20, 21] and other authors considered and investigated (weighted or non-weighted) classes of L^p-integrable holomorphic functions and obtained reproducing kernels in an explicit form. In particular, M. M. Djrbashian (1945, 1948) introduced weighted L^p-classes $H^p_\alpha(\mathbb{D})$ of holomorphic functions in $\mathbb{D}$ with weight functions $(1-|\zeta|^2)^\alpha, \zeta \in \mathbb{D}$ $(1 \le p < \infty, \alpha > -1)$, and established weighted integral representations for these classes. These representations had been successfully applied in the initial version of his theory of factorization of meromorphic functions in the unit disc (see [20, 21]; for other version of factorization theory see [22]). Only 30-40 years later a great interest to these spaces arose and weighted integral representations step by step started to play an important role in theory of functions in one and several complex variables. Several important investigations should be mentioned: L. K. Hua [23], F. Forelli and W. Rudin [24], W. Rudin [25], M. M. Djrbashian [26, 27], M. M. Djrbashian and V.

S. Zakaryan [28], F. A. Shamoyan [29, 30], A. E. Djrbashian and F. A. Shamoyan [31], H. Hedenmalm, B. Korenblum and K. Zhu [32], P. Duren and A. Schuster [33], K. Zhu [34], A. M. Jerbashian and K. L. Avetisyan [35], A. M. Jerbashian [36, 37], A. I. Petrosyan [38], K. L. Avetisyan [39], M. M. Djrbashian and A. H. Karapetyan [40] - [44], A. H. Karapetyan [45].

Assume that $1 < p < +\infty, \rho > 0, \alpha > -1, \gamma > -1$. In [46] M. M. Djrbashian introduced weighted L^p-classes $H^p_{\alpha,\rho,\gamma}(\mathbb{D}) \equiv L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \cap H(\mathbb{D})$ of holomorphic functions in $\mathbb{D}$ with weight functions $(1-|\zeta|^{2\rho})^\alpha \cdot |\zeta|^{2\gamma}, \zeta \in \mathbb{D}$ $(1 \le p < \infty, \alpha > -1)$, and for these classes established analogs of his previous (1945, 1948) weighted integral representations, but this time written out by means of special Mittag-Leffler type reproducing kernels

$$S_{\alpha,\rho,\gamma}(z,\zeta) = \frac{\rho}{\pi \cdot \Gamma(\alpha+1)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(\mu+\alpha+1+\frac{k}{\rho})}{\Gamma(\mu+\frac{k}{\rho})} \cdot z^k \cdot \overline{\zeta}^k, \quad \mu = \frac{1+\gamma}{\rho},$$

adapted to new weight functions.

Note that for $\rho = 1, \gamma = 0$ the kernels above coincides with the classical M. M. Djrbashian's [20, 21] kernels $(\alpha+1)/\pi \cdot (1-z\overline{\zeta})^{-(2+\alpha)}$.

In the thesis in the formula above we take complex parameters β and φ instead of α and γ respectively. As a result, we obtain a whole family of kernels $S_{\beta,\rho,\varphi}(z,\zeta)$ and prove that these kernels have a reproducing property for the spaces $H^p_{\alpha,\rho,\gamma}(\mathbb{D})$. Moreover, for the similar spaces $h^p_{\alpha,\rho,\gamma}(\mathbb{D})$ of complex-valued harmonic functions we also reveal a family of reproducing kernels written out by means of $S_{\beta,\rho,\varphi}(z,\zeta)$.

In [47] D. Pompeiu (1904) presented a generalization of the Cauchy integral formula for smooth functions:

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint_{\Omega} \frac{\frac{\partial f(\zeta)}{\partial \overline{\zeta}}}{\zeta - z} dm(\zeta), \quad z \in \Omega,$$

where Ω is a bounded domain with piecewise-smooth boundary, $f \in C^1(\overline{\Omega})$ and m is two-dimensional Lebesgue measure in the complex plane.

Later on appeared a new formula which for the case of the unit disc $\mathbb{D}$ unified both M. M. Djrbashian's and D. Pompeiu's formulas in the following sense ($Re\beta > -1$):

$$f(z) = \frac{\beta+1}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)(1-|\zeta|^2)^\beta}{(1-z\overline{\zeta})^{2+\beta}} dm(\zeta) - \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\partial f(\zeta)/\partial\overline{\zeta}}{\zeta - z} \cdot \left(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}.$$

This representation follows from the investigations of

• Ph. Charpentier [48], if $f \in C^1(\overline{\mathbb{D}})$;

• F. A. Shamoyan [29], if $f \in C^1(\mathbb{D}), grad(f) \in L^1(\mathbb{D})$, and β is real;

• A. I. Petrosyan [49, 50], if $1 \le p < \infty, \alpha > -1, f \in C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D}), \partial f(\zeta)/\partial\overline{\zeta} \in L^p_\alpha(\mathbb{D})$ and $\beta = \alpha$;

• A. H. Karapetyan [51, 52], if $1 \le p < \infty, \alpha > -1, f \in C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D}), \partial f(\zeta)/\partial\overline{\zeta} \in L^p_{\alpha+1}(\mathbb{D})$, and $Re\beta \ge \alpha$.

Note that integral representations of such type were obtained in [48] for the unit ball B_n and the unit polydisc U^n and later were generalized in [49, 50] (for U^n), in [51] (for the matrix unit disc) and in [53] (where very general weight functions were considered for B_n).

In [46] a further generalization of the last formula was proved by taking a weight function of the type $|\zeta|^{2\gamma} \cdot (1 - |\zeta|^{2\rho})^\alpha$ instead of $(1 - |\zeta|^2)^\alpha$ ($\rho > 0$, $\alpha > -1$ and $\gamma > -1$). More precisely, the following representation holds:

$$f(z) = \iint\limits_{\mathbb{D}} f(\zeta) \cdot S_{\alpha,\rho,\gamma}(z;\zeta) \cdot (1 - |\zeta|^{2\rho})^\alpha \cdot |\zeta|^{2\gamma} dm(\zeta)$$
$$- \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\partial f(\zeta)/\partial\overline{\zeta}}{\zeta - z} \cdot Q_{\alpha,\rho,\gamma}(z;\zeta) dm(\zeta), \qquad z \in \mathbb{D},$$

where $f \in C^1(\overline{\mathbb{D}})$, for $S_{\alpha,\rho,\gamma}(z;\zeta)$ see above and $Q_{\alpha,\rho,\gamma}(z;\zeta)$ is written out explicitly in an integral form by means of $S_{\alpha,\rho,\gamma}(z;\zeta)$.

In [46] it was proved that for $\rho = 1, \gamma = 0$, the kernel $Q_{\alpha,\rho,\gamma}(z;\zeta)$ coincides with $\big((1 - |\zeta|^2)/(1 - z\overline{\zeta})\big)^{\alpha+1}$.

As it follows from [50], where multidimensional analogue of this result was obtained, the restrictive condition $f \in C^1(\overline{\mathbb{D}})$ in the last formula can be replaced by

$$f \in C^1(\mathbb{D}) \cap L^p_{\alpha,\rho,\gamma}(\mathbb{D}), \qquad \frac{\partial f(\zeta)}{\partial\overline{\zeta}} \in L^p_{\alpha,\rho,\gamma}(\mathbb{D}), \quad (1 \le p < \infty).$$

In the thesis we weaken the second growth condition by assuming that

$$\frac{\partial f(\zeta)}{\partial\overline{\zeta}} \in L^p_{\alpha+1,\rho,\gamma}(\mathbb{D})$$

and under the new conditions establish (using another approach in comparison with [50]) a family of Pompeiu type representations by means of kernels $S_{\beta,\rho,\varphi}(z;\zeta)$ and $Q_{\beta,\rho,\varphi}(z;\zeta)$, where β and φ are complex parameters.

Lets go back to D. Pompeiu formula. As the first summand in it is holomorphic in Ω, we can conclude that a solution of so-called $\overline{\partial}$-equation

$$\frac{\partial g(z)}{\partial \overline{z}} = v(z), \quad z \in \Omega,$$

with given $v \in C^1(\Omega)$ and unknown $g \in C^1(\Omega)$ can be found in the form

$$g(z) = -\frac{1}{\pi} \cdot \iint\limits_{\Omega} \frac{v(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \Omega.$$

In general, the $\overline{\partial}$-equation plays an important role in complex analysis (especially in several complex variables). Nevertheless, in one complex variable it also had important applications in the corona problem solution (see [54]) and approximation theory (see, for example, [55]).

In direction of solving $\overline{\partial}$-equation it should be mentioned that the last formula solves $\overline{\partial}$-equation in the cases when $v \in C_c^k(\Omega)$ (see [56, Theorem 1.2.2]), i.e. $v \in C^k(\Omega)$ and v has a compact support in Ω, or, more general, $v \in C^k(\Omega) \cap L^\infty(\Omega)$ or $v \in C^k(\Omega) \cap L^1(\Omega)$ (see [25, Proposition 16.3.2] and [57, Theorem 1.1.3]) ($k = 1, 2, 3, ..., \infty$).

Taking into account the above-mentioned generalizations of Pompeiu's formula we can take as a weighted version of the last formula the following one:

$$g_\beta(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z} \cdot \left(\frac{1 - |\zeta|^2}{1 - z\overline{\zeta}}\right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D},$$

As it follows from the multidimensional results of [48], if $v \in C^1(\mathbb{D}) \cap L^p_{\alpha+1}(\mathbb{D})$, then the function $g_\beta \in C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D})$, satisfies the $\overline{\partial}$-equation and, moreover,

$$\|g_\beta\|_{p,\alpha} \leq const(\alpha, \beta) \cdot \|v\|_{p,\alpha+1}.$$

In the thesis we replace the kernel $\left((1 - |\zeta|^2)/(1 - z\overline{\zeta})\right)^{\beta+1}$ by $Q_{\beta,\rho,\varphi}(z,\zeta)$ and, assuming that $v \in C^k(\mathbb{D}) \cap L^p_{\alpha+1,\rho,\gamma}(\mathbb{D})$, obtain a family $g_{\beta,\rho,\varphi}(z)$ of C^k-solutions of the $\overline{\partial}$-equation in the unit disc. For $p=2$, $\beta=\alpha$, $\varphi=\gamma$ and under the condition $v(\zeta)\in C^k(\mathbb{D})\cap L^p_{\alpha,\rho,\gamma}(\mathbb{D})$ our result follows from [49, 50].

Pompeiu type formulas are of great interest in the case of unbounded domains too. In the case of the upper half-plane Π_+ the following formula is true, which is a consequence of the corresponding multidimensional result [58, Theorem2.2]:

$$f(w)=\frac{2^\beta(\beta+1)}{\pi}\iint\limits_{\Pi_+}\frac{f(\eta)(Im\eta)^\beta}{[i(\overline{\eta}-w)]^{2+\beta}}dm(\eta)$$
$$-\frac{2^{\beta+1}}{\pi}\iint\limits_{\Pi_+}\frac{\partial f(\eta)/\partial\overline{\eta}}{\eta-w}\cdot\frac{(Im\eta)^{\beta+1}}{[i(\overline{\eta}-w)]^{\beta+1}}dm(\eta),\quad w\in\Pi_+,$$

where the spaces $L^p_{\alpha,\gamma}(\Pi_+)$ are generated by the norms

$$\|f\|_{p,\alpha,\gamma}\equiv\left(\int\limits_{\Pi_+}\frac{|f(\eta)|^p\cdot(Im\eta)^\alpha}{|\eta+i|^\gamma}dm(\eta)\right)^{1/p}.$$

Note that in the upper half-plane the spaces of such type were considered in [59]. In the case of holomorphic functions, when the second summand disappears, the this representation was obtained in [60], [61].

Similar to the case of unit disc, it is reasonable to expect that the second summand of the representation above can generate a formula for a solution of the $\overline{\partial}$-equation

$$\frac{\partial f(w)}{\partial\overline{w}}=u(w),\qquad w\in\Pi_+.$$

In the thesis it is shown that this expectation takes place, i.e. the formula generated by the second summand of above mentioned weighted integral representation in Π_+ gives a family of solutions of $\overline{\partial}$-equation under the certain restrictions on the function u.

In connection with solution of the $\overline{\partial}$-equation in Π_+ several publication should be mentioned. In [62] solutions were constructed under the assumption that its right-hand sides are complex Carleson measures. These solutions were written out in a form of non-linear integral operators and were understood in the sense of distributions.

In [63], [64] for the domain called "future tube" (a multidimensional analogue of Π_+), under the assumption that the right hand-side has a bounded support, solutions of $\overline{\partial}$-equation were constructed and uniform estimates of these solutions inside the domain were obtained.

Goals.

- To obtain an estimate for the integral logarithmic means of Blaschke products by it's zeros counting function in the lower half-plane.
- To obtain necessary and sufficient condition for the boundedness of the integral logarithmic means of Blaschke products in the lower half-plane (in the case when zeros have a special distribution).
- To constract a Blaschke product of a given quantity index in the lower half-plane.
- To introduce a family of reproducing kernels in the unit disc (depending on complex parameters) and investigate their growth near the boundary of the unit disc as well as their holomorphic dependence in complex parameters.
- To obtain a family of weighted integral representations of some weighted L^p-classes of holomorphic and harmonic functions in the unit disc by means of introduced reproducing kernels.
- To obtain a family of weighted $\overline{\partial}$-integral representations for some weighted L^p-classes of C^1-functions in the unit disc.
- To obtain a family of weighted solutions of $\overline{\partial}$-equation in the unit disc.
- To obtain a family of weighted solutions of $\overline{\partial}$-equation in the upper half-plane and establish weighted norm estimates for the mentioned solutions.

Research methods. Method of Fourier transforms for meromorphic functions and methods of complex analysis are used.

Scientific novelty. All results of the thesis are new. They are generalizations or continuations of known results.

Theoretical and practical value. The thesis is a theoretical investigation. Obtained results can be applied to problems of function theory as well as can be generalized for functions in several complex variables.

Approbation of results. The main results of the thesis were reported to "YSU SSS 5th international conference" (April 2018), "Conference Dedicated to the Memory of Academician Mkhitar Djrbashyan" (October 2018), "First Analysis Mathematica International Conference (ANMath) 2019" (August 2019), seminar of the section of Complex Analysis of the Institute of Mathematics of NAS RA.

Publications. The main results of the thesis are published in the author's works [75] - [81].

Structure and volume of the thesis. The thesis consists of introduction, four chapters, conclusion and references containing 83 items. Total number of pages is 85.

The content of the work

Introduction contains a brief historical review, the main preceding results and brief description of obtained results.

In the chapter 1 we consider the infinite Blaschke products in the lower half-plane $G = \{w : Im\,(w) < 0\}$ which have the form

$$B\,(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w - \overline{w}_k}, \tag{1}$$

where the sequence of complex numbers $\{w_k\}_1^\infty = \{u_k + iv_k\}_1^\infty$ lies in G and

$$\sum_{k=1}^{\infty} |v_k| < +\infty. \tag{2}$$

The last condition ensures the convergence of the corresponding infinite product to an analytic function in G with zeros $\{w_k\}_1^\infty$.

Define an integral logarithmic mean of order q, $1 \leq q < +\infty$, of Blaschke products for the half-plane by the formula

$$m_q\,(v, B) = \left(\int_{-\infty}^{+\infty} |\log |B\,(u + iv)||^q \,du \right)^{\frac{1}{q}}, \qquad -\infty < v < 0. \tag{3}$$

Let's denote by $n(v)$ the number of zeros of the function B in the half-plane $\{w : Im\,(w) \leq v\}$.

We obtain estimates for $m_q\,(v, B)$ by the function $n\,(v)$ $(1/p + 1/q = 1)$:

Theorem 1.1. *a) In the case $q = 1$ we have*

$$m_1\,(v, B) = \int_{-\infty}^{+\infty} |\log |B\,(u + iv)|| \,du = \sqrt{2\pi} \int_{v}^{0} n\,(t)\,dt. \tag{4}$$

b) In the case $1 < q < +\infty$ there exists a constant c_p such that

$$m_q\,(v, B) \leq c_p\,|v|^{-\frac{1}{p}} \int_{v}^{0} n\,(t)\,dt, \qquad -\infty < v < 0. \tag{5}$$

Corollary 1.1. *If $1 \leq q < +\infty$ and for some $0 < \alpha < 1$*

$$n\,(v) = O\left(|v|^{-\alpha}\right),\ v \to 0,$$

then $m_q(v,B) = O\left(|v|^{\frac{1}{q}-\alpha}\right), \ v \to 0.$

When the sequence $\{w_k\}$ of the zeros of the Blaschke product B has a special distribution we arrive at the following important assertion:

Theorem 1.2. *If the sequence* $\{w_k\}_1^\infty$ *belongs to a vertical ray* $\{w = u_0 + ih : -\infty < h < 0\}$ *and* $1 < q \leq 2$, *then the boundedness of the function* $m_q(v,B)$ *is equivalent to the condition*

$$n(v) = O\left(|v|^{-\frac{1}{q}}\right), v \to 0.$$

Let us define (following the [5]) the quantity index $p(B)$ of Blaschke product for the lower half-plane as

$$p(B) = p - 1,$$

where $p^{-1} + q^{-1} = 1$ and $q = \sup\{s \in [1,\infty) : m_s(v,B) = O(1), v \to 0\}$.

Based on the method developed in [11] we obtain a complete analogue of the corresponding result [5] related to the unit disc:

Theorem 1.3. *For any* $p \in [1,+\infty]$ *there exists a Blaschke product for the lower half-plane of given quantity index* $p(B) = p - 1$.

In the case of a half-plane the problem of the connection of the boundedness of $m_2(v,\pi_\alpha)$ with the zeros distribution of A.M. Jerbashian's (see [10]) products π_α was solved by the method of Fourier transforms for meromorphic functions (see [11]). The function π_α coincides with B for $\alpha = 0$.

In the chapter 2 we consider the properties of the spaces $H^p_{\alpha,\rho,\gamma}(\mathbb{D})$ and $h^p_{\alpha,\rho,\gamma}(\mathbb{D})$ and establish their weighted integral representations. As it was mentioned above we extend the definition of the kernel $S_{\alpha,\rho,\gamma}$ introduced by M.M. Djrbashian [46] by replacing α and γ by complex parameters β and φ respectively.

Definition 2.4. *Let* $\rho > 0, Re\beta > -1, Re\varphi > -1$ *and* $\mu = \dfrac{1+\varphi}{\rho}$. *Then put*

$$S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi \cdot \Gamma(\beta+1)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(\mu+\beta+1+\frac{k}{\rho})}{\Gamma(\mu+\frac{k}{\rho})} \cdot z^k \cdot \overline{\zeta}^k, \tag{6}$$

where $z \in \mathbb{D}$ *and* $\zeta \in \overline{\mathbb{D}}$.

We state the main properties of the kernel introduced including: growth estimates near the boundary of the unit disc, holomorphic dependence in z and antiholomorphic dependence in ζ, integral representation by means of Mittag-Leffler type function, holomorphic dependence in complex parameters β and φ:

Theorem 2.1. *Assume $\rho > 0, Re\beta > -1, Re\varphi > -1, \mu = \dfrac{1+\varphi}{\rho}$, $z \in \mathbb{D}, \zeta \in \overline{\mathbb{D}}$ and the kernel $S_{\beta,\rho,\varphi}(z,\zeta)$ is defined by (6).*

(a) If

$$-1 < a_1 \leq Re\beta \leq a_2 < +\infty, \qquad |Im\beta| \leq A < +\infty, \tag{7}$$

$$-1 < b_1 \leq Re\varphi \leq b_2 < +\infty, \qquad |Im\varphi| \leq B < +\infty \tag{8}$$

and $\zeta \in \overline{\mathbb{D}}$, $|z| \leq \lambda < 1$, then the expression of $S_{\beta,\rho,\varphi}$ can be uniformly majorated by the convergent series

$$const(a_1, a_2, A, b_1, b_2, B, \rho) \cdot \sum_{k=0}^{\infty} (k+1)^{a_2+1} \lambda^k. \tag{9}$$

In particular the kernel $S_{\beta,\rho,\varphi}(z,\zeta)$ is well-defined by (6). Moreover,

$$\begin{aligned} |S_{\beta,\rho,\varphi}(z,\zeta)| &\leq \frac{const(a_1, a_2, A, b_1, b_2, B, \rho)}{(1-|z|\cdot|\zeta|)^{2+Re\beta}} \\ &\leq \frac{const(a_1, a_2, A, b_1, b_2, B, \rho)}{(1-|z|)^{2+Re\beta}} \leq \frac{const(a_1, a_2, A, b_1, b_2, B, \rho)}{(1-\lambda)^{2+a_2}}. \end{aligned} \tag{10}$$

(b) For all $z \in \mathbb{D}$ and for all $\zeta \in \overline{\mathbb{D}}$

$$S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi \cdot \Gamma(\beta+1)} \cdot \int_0^{\infty} e^{-t} \cdot t^{\mu+\beta} \cdot E_\rho\left(t^{\frac{1}{\rho}} z\overline{\zeta}; \mu\right) dt. \tag{11}$$

Moreover, the function under the sign of the integral is majorated by a positive integrable function uniformly in $z \in K \subset \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$, where K is a compact set.

Corollary 2.5. *For fixed $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$ the kernel $S_{\beta,\rho,\varphi}(z,\zeta)$ is holomorphic in β and φ with $Re\beta > -1$ and $Re\varphi > -1$. If β and φ are fixed, then for a fixed $\zeta \in \overline{\mathbb{D}}$, $S_{\beta,\rho,\varphi}(z,\zeta)$ is holomorphic in $z \in \mathbb{D}$ and for a fixed $z \in \mathbb{D}$, $S_{\beta,\rho,\varphi}(z,\zeta)$ is antiholomorphic in $\zeta \in \mathbb{D}$ and continuous in $\zeta \in \overline{\mathbb{D}}$.*

The main results of this chapter relate to weighted integral representations of introduced spaces of holomorphic and harmonic functions and can be formulated as follows:

Theorem 2.3. *Assume that* $1 \le p < +\infty, \rho > 0, \alpha > -1, \gamma > -1$, *complex numbers* β *and* φ *satisfy the following conditions*

$$\begin{cases} Re(\beta) \ge \alpha, Re(\varphi) \ge \gamma, \quad p = 1 \\ \begin{cases} Re(\beta) > \frac{\alpha+1}{p} - 1 \\ Re(\varphi) > \frac{\gamma+1}{p} - 1 \end{cases}, \quad 1 < p < \infty \end{cases} \tag{12}$$

and $\mu = \frac{\varphi+1}{\rho}$.

Then each function $f \in H^p_{\alpha,\rho,\gamma}(\mathbb{D})$ *has the following representations:*

$$f(z) = \iint\limits_{\mathbb{D}} f(\zeta) \cdot S_{\beta,\rho,\varphi}(z,\zeta) \cdot (1 - |\zeta|^{2\rho})^{\beta} \cdot |\zeta|^{2\varphi} dm(\zeta), \qquad z \in \mathbb{D}, \tag{13}$$

and

$$\overline{f(0)} = \iint\limits_{\mathbb{D}} \overline{f(\zeta)} \cdot S_{\beta,\rho,\varphi}(z,\zeta) \cdot \left(1 - |\zeta|^{2\rho}\right)^{\beta} \cdot |\zeta|^{2\varphi} dm(\zeta), \qquad z \in \mathbb{D}. \tag{14}$$

Theorem 2.5. *Assume* $1 \le p < \infty$, $\rho > 0$, $\alpha > -1, \gamma > -1$,

$$\begin{cases} Re(\beta) \ge \alpha, Re(\varphi) \ge \gamma, \quad p = 1 \\ \begin{cases} Re(\beta) > \frac{\alpha+1}{p} - 1 \\ Re(\varphi) > \frac{\gamma+1}{p} - 1 \end{cases}, \quad 1 < p < \infty \end{cases}$$

and $\mu = \dfrac{\varphi + 1}{\rho}$. *For each* $u \in h^p_{\alpha,\rho,\gamma}(\mathbb{D})$ *the following representation holds:*

$$u(z) = \iint\limits_{\mathbb{D}} u(\zeta) \cdot \left(S_{\beta,\rho,\varphi}(z,\zeta) + S_{\beta,\rho,\varphi}(\zeta,z) - \frac{\rho}{\pi} \cdot \frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\cdot\Gamma(\mu)} \right)$$
$$\cdot (1 - |\zeta|^{2\rho})^{\beta} |\zeta|^{2\varphi} dm(\zeta), \qquad z \in \mathbb{D}. \tag{15}$$

Remark. *In [65]-[74] one can find various interesting results relating to weighted integral representations of harmonic functions.*

In the chapter 3 for weighted L^p-spaces of C^1-functions in the unit disc we obtain a family of Pompeiu type weighted integral representations with kernels $S_{\beta,\rho,\varphi}(z;\zeta)$ (see (6)) and $Q_{\beta,\rho,\varphi}(z;\zeta)$ (see definition below).

Definition. *Let $\rho > 0, Re\beta > -1, Re\varphi > -1$ and $\mu = \dfrac{1+\varphi}{\rho}$. Then for $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$ put*

$$Q_{\beta,\rho,\varphi}(z;\zeta) = 1 + \frac{(z-\zeta)\rho}{\zeta \cdot \Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \frac{z^k}{\zeta^k} \int_0^{|\zeta|^2} (1-t^\rho)^\beta \, t^{\varphi+k} dt \qquad (16)$$

if $\zeta \in \overline{\mathbb{D}} \backslash \{0\}$ and

$$Q_{\beta,\rho,\varphi}(z;0) \equiv 1. \qquad (17)$$

Note that in difference to with [46] and [49, 50], we write out the kernels $Q_{\beta,\rho,\varphi}(z;\zeta)$ in a series form which admits to specify their certain properties.

One of the main results of this chapter is formulated as follows:

Theorem 3.1. *Let $1 \le p < +\infty$, $\alpha > -1, \gamma > -1, \rho > 0, Re\beta \ge \alpha$, and $Re\varphi \ge \gamma$. Also, let*

$$f \in L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \cap C^1(\mathbb{D}) \qquad (18)$$

and

$$\frac{\partial f(\zeta)}{\partial \overline{\zeta}} \in L^p_{\alpha+1,\rho,\gamma}(\mathbb{D}). \qquad (19)$$

Then for every $z \in \mathbb{D}$

$$f(z) = \iint_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z;\zeta) \left(1-|\zeta|^{2\rho}\right)^\beta |\zeta|^{2\varphi} dm(\zeta)$$

$$- \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\partial f(\zeta)/\partial\overline{\zeta}}{\zeta - z} Q_{\beta,\rho,\varphi}(z;\zeta) dm(\zeta). \qquad (20)$$

Remark. *Note that for holomorphic functions* (20) *coincides with the representation* (13).

The second summand of the integral representation (20) inspires to put formally

$$g_{\beta,\rho,\varphi}(z) = -\frac{1}{\pi} \cdot \iint_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z} \cdot Q_{\beta,\rho,\varphi}(z;\zeta) dm(\zeta), \quad z \in \mathbb{D}, \qquad (21)$$

for arbitrary $v \in C^k(\mathbb{D})$ with expectations that $g_{\beta,\rho,\varphi}(z)$ can be a solution of $\overline{\partial}$-equation in the unit disc under certain assumptions on the function v ($\rho > 0$, $Re\beta > -1$, $Re\varphi > -1$ and $\mu = (\varphi + 1)/\rho$).

The following two results state the assumptions on a function v under which the function $g_{\beta,\rho,\varphi}$ will be a C^k-solution of the $\overline{\partial}$-equation.

Theorem 3.2. *If $v \in C^k_c(\mathbb{D})$, $k = 1, 2, 3, \ldots, \infty$, then $g(z) \equiv g_{\beta,\rho,\varphi}(z)$ is of class $C^k(\mathbb{D})$ and satisfies the $\overline{\partial}$-equation.*

Theorem 3.3. *Assume that $\alpha > -1$, $\gamma > -1$, $1 \leq p < +\infty$, $Re\beta \geq \alpha$ and $Re\varphi \geq \gamma$. Also let*

$$v(\zeta) \in C^k(\mathbb{D}) \cap L^p_{\alpha+1,\rho,\gamma}(\mathbb{D}) \tag{22}$$

for $k = 1, 2, 3, \ldots, \infty$. Then $g(z) \equiv g_{\beta,\rho,\varphi}(z)$ is of class $C^k(\mathbb{D})$ and satisfies the $\overline{\partial}$-equation.

In the chapter 4 we formally introduce (based on the formula obtained in [58, Theorem2.2]) an integral operator

$$T^*_\beta(u)(w) = -\frac{2^{\beta+1}}{\pi} \cdot \iint\limits_{\Pi_+} \frac{u(\eta)}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta} - w))^{\beta+1}} dm(\eta), \quad w \in \Pi_+, \tag{23}$$

where $Re\beta > -1$ and $u(\eta), \eta \in \Pi_+$, is, in general, a complex-valued measurable function.

Using well-known Cayley transforms we connect this operator with the corresponding integral operator

$$T_\beta(v)(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z} \cdot \left(\frac{1 - |\zeta|^2}{1 - z\overline{\zeta}} \right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D},$$

where $Re\beta > -1$ and $v(\zeta), \zeta \in \mathbb{D}$, is, in general, a complex-valued measurable function (see also the definition of $g_\beta(z)$ above). Remember that this operator gives weighted solutions of the $\overline{\partial}$-equation in the unit disc (see [48] and [79]). As a result, we obtain various restrictions on the function u under which $T^*_\beta(u)$ gives a solution of the $\overline{\partial}$-equation

$$\frac{\partial f(w)}{\partial \overline{w}} = u(w), \qquad w \in \Pi_+. \tag{24}$$

Moreover, weighted L^p-estimates for the operator T^*_β are obtained. The main results of the chapter can be formulated as follows:

Theorem 4.3. *Let* $u \in C^k(\Pi_+)$, $k = 1, 2, 3, \ldots, \infty$, $Re\beta > -1$ *and*

$$\frac{|u(\eta)| \cdot (Im\eta)^{Re\beta+1}}{|\eta + i|^{Re\beta+2}} \in L^1(\Pi_+). \tag{25}$$

Put $f_\beta(w) \equiv T^*_\beta(u)(w)$, $w \in \Pi_+$, *then* $f_\beta(w) \in C^k(\Pi_+)$ *and* (24) *is true.*

Teorem 4.4. *Let a function* $u \in C^k(\Pi_+)$ $(k = 1, 2, 3, \ldots, \infty)$ *and satisfies one of the following conditions:*

(a) $u(\eta) \cdot Im\eta \in L^1_{\alpha,\gamma}(\Pi_+)$, $\alpha > -1, \gamma \leq 2 + \alpha, Re\beta \geq \alpha$,

(b) $u(\eta) \cdot Im\eta \in L^p_{\alpha,\gamma}(\Pi_+)$, $1 < p < \infty, \alpha > -1, \gamma < 2 + \alpha, Re\beta > \frac{\alpha+1}{p} - 1$. *Put* $f_\beta(w) \equiv T^*_\beta(u)(w)$, $w \in \Pi_+$, *then* $f_\beta(w) \in C^k(\Pi_+)$ *and* (24) *is true.*

Theorem 4.5. *Let* $1 \leq p < \infty$, $\alpha > -1$, $\gamma \in \mathbb{R}$. *Assume that the conditions*

$$4p + pRe\beta + \gamma - 4 - 2\alpha \geq 0 \tag{26}$$

and

$$Re\beta > 4p + pRe\beta + \gamma - 5 - \alpha > -1 \tag{27}$$

take place. Then for an arbitrary function $u \in C^k(\Pi_+) \cap L^p_{\alpha,\gamma}(\Pi_+)$, $k = 1, 2, 3, \ldots, \infty$, *the integral operator* T^*_β *solves the corresponding* $\overline{\partial}$*-equation in* Π_+, *i.e. for the function* $f_\beta(w) \equiv T^*_\beta(u)(w)$, $w \in \Pi_+$, (24) *holds and* $f_\beta(w) \in C^k(\Pi_+)$. *Moreover, the following estimate is true:*

$$\|f_\beta\|_{p,\ 4p+pRe\beta+\gamma-5-\alpha,\ 6p+pRe\beta+2\gamma-6-2\alpha} \leq const(p, \beta, \alpha, \gamma) \cdot \|u\|_{p,\alpha,\gamma}. \tag{28}$$

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Ամփոփում

Ատենախոսությունում ստացվել են հետևյալ հիմնական արդյունքները:

- Հետազոտվել է կոմպլեքս հարթության ստորին կիսահարթությունում Բլյաշկեի դասական արտադրյալների վարքագիծը: Կիսահարթության եզրի մոտ ստացվել են կամայական կարգի ինտեգրալ լոգարիթմական միջինների աճի գնահատականներ մերոմորֆ ֆունկցիաների համար Ֆուրիեի ձևափոխության մեթոդի կիրառմամբ: Ապացուցվել է նախապես տրված քանակական ինդեքս ունեցող Բլյաշկեի արտադրյալի գոյությունը:

- Միավոր շրջանում հոլոմորֆ կամ հարմոնիկ ֆունկցիաների կշռային L^p դասերի համար $(1 \leq p < \infty)$ ստացվել են կշռային ինտեգրալ ներկայացումներ $(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}$ տիպի կշռային ֆունկցիաներով և Միտտագ-Լեֆֆլերի տիպի վերարտադրող կորիզների միջոցով: Մանրամասնորեն հետազոտվել են նշված կորիզների հատկությունները:

- Միավոր շրջանում $|\zeta|^{2\gamma}(1-|\zeta|^{2\rho})^{\alpha}$ տիպի կշռային ֆունկցիաներով կշռային L^p տարածություններին $(1 \leq p < \infty)$ պատկանող C^1 դասի ֆունկցիաների համար ստացվել են $f = P(f) - T(\overline{\partial} f)$ տիպի կշռային $\overline{\partial}$-ինտեգրալ ներկայացումներ:

- Միավոր շրջանում դիտարկվել է $\partial g(z)/\partial \overline{z} = v(z)$ հավասարումը: Ենթադրելով, որ v-ն C^k դասի ֆունկցիա է $|z|^{2\gamma}(1-|z|^{2\rho})^{\alpha}$ տիպի կշռային ֆունկցիաներով կշռային L^p տարածություններից $(1 \leq p < \infty)$, կառուցվել է C^k դասի g լուծումների մի ամբողջ ընտանիք:

- Վերին կիսահարթությունում դիտարկվել է $\partial f(w)/\partial \overline{w} = u(w)$ հավասարումը: Ենթադրելով, որ u-ն C^k դասի ֆունկցիա է $(Imw)^{\alpha}|w+i|^{-\gamma}$ տիպի կշռային ֆունկցիաներով կշռային L^p տարածություններից $(1 \leq p < \infty)$, կառուցվել է C^k դասի f լուծումների մի ամբողջ ընտանիք: Ավելին, f ֆունկցիայի կշռային նորմերը գնահատվել են u ֆունկցիայի կշռային նորմերով:

Заключение

- Изучено поведение классических произведений Бляшке в нижней полуплоскости комплексной плоскости. Применяя метод преобразования Фурье для мероморфных функций, получены оценки роста интегральных логарифмических средних произвольного порядка вблизи границы полуплоскости. Доказано существование произведения Бляшке с заранее заданным значением индекса величины.

- Для голоморфных или гармонических в единичном круге функций из весовых L^p-классов $(1 \leq p < \infty)$ получены весовые интегральные представления с весовыми функциями типа $(1 - |\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}$ и воспроизводящими ядрами типа Миттаг-Леффлера. Свойства этих ядер основательно изучены.

- Для C^1-функций в единичном круге из весовых L^p-классов $(1 \leq p < \infty)$ с весовыми функциями типа $(1 - |\zeta|^{2\rho})^{\alpha}|\zeta|^{2\gamma}$ получены весовые $\overline{\partial}$-интегральные представления типа $f = P(f) - T(\overline{\partial} f)$.

- В единичном круге рассмотрено уравнение $\partial g(z)/\partial \overline{z} = v(z)$. Полагая, что v суть функция класса C^k из весовых L^p-пространств $(1 \leq p < \infty)$ с весовыми функциями типа $|z|^{2\gamma}(1 - |z|^{2\rho})^{\alpha}$, строится целое семейство решений g класса C^k.

- В верхней полуплоскости рассмотрено уравнение $\partial f(w)/\partial \overline{w} = u(w)$. Полагая, что u суть функция класса C^k из весовых L^p-пространств $(1 \leq p < \infty)$ с весовыми функциями типа $(Imw)^{\alpha}|w + i|^{-\gamma}$, строится целое семейство решений f класса C^k. Более того, весовые нормы функции f оценены через весовые нормы функции u.