REFERENCE

to the Candidate Thesis of ***Feliks Hayrapetyan***
"Weighted Integral Representations and Properties of Some Weighted Classes of Holomorphic, Harmonic and Smooth Functions in the Unit Disc and Half-Plane"
submitted for the degree of Candidate of Physical and Mathematical Sciences
A.01.01-Mathematical Analysis

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The Thesis can be conditionally divided into two parts: Chapter 1 and Chapters 2,3,4.

Chapter 1 establishes some half-plane analogs of the solution of a problem in the unit disc, posed by A.Zygmund, also it solves some more general problem.

Chapters 2,3 are devoted to proving some very general Pompeu type representations of functions from several weighted spaces of functions holomorphic, harmonic or even just measurable in the unit disc $\mathbb{D}$ and in the half-plane $\prod_+$ of the complex plane, which are used in Chapter 4 for finding some new solutions of the $\overline{\partial}$-equation which is significant in Complex Analysis.

In the below more detailed description of the mentioned chapters, the references correspond to those in Thesis, while the numeration of theorems correspond to that in its Introduction.

Chapter 1. It is well-known that the estimation problem of the square order logarithm integral means of the Blaschke product in the unit disc of the complex plane by means of the counting function of its zeros was first posed by A.Zygmund. In 1969, using a Fourier series method this problem was solved by G.R.MacLane and L.A.Rubel [1]. Later, V.V.Eiko and A.A.Kondratyuk [2] investigated the more general problem in the case of any order $1 \leq q < +\infty$.

Aimed at solution of the latter problem in the lower half-plane, G.V.Mikayelyan's [9, 11, 12] method of *Fourier transforms for meromorphic functions* is used to investigate the growth of the Blaschke products in the half-plane and to obtain the natural half-plane analogs of the mentioned results in the disc.

Considering the Blaschke product of the form

$$B(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w - \overline{w}_k}, \quad \sum_{k=1}^{\infty} |\mathrm{Im}\, w_k| < +\infty,$$

in the lower half-plane and denoting

$$m_q(v, B) = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\log|B(u+iv)|\right|^q\right)^{1/q}, \quad v < 0, \quad 1 < q < +\infty.$$

in Theorem 0.7 the author could prove the estimate

$$m_p(v,B) \le C_p|v|^{-1/p}\int_v^0 n(t)dt, \quad -\infty < v < 0,$$

where p is the conjugate index of q, i.e. $1/p + 1/q = 1$, C_p is a constant depending solely on p and $n(t)$ is the quantity of zeros w_k of $B(w)$, such that $\operatorname{Im} w_k \le t$. A more delicate result is established in the particular case when the zeros are on the imaginary axis, also it is proved a half-plane analog of a result by M.O.Ghirnyk and A.A.Kondratyuk [2] on the existence of Blaschke products with a given so-called *quantity index*. ?

In my opinion, one of the most interesting continuations of the mentioned problems is the consideration of some Hardy type spaces of Blaschke ω-products or Green type ω-potentials, which is done for $q = 2$ in the recent works[1, 2] with some of my Colombian pupils. Thus, a possible further continuation of the work of Feliks Hayrapetyan can be a consideration of more general, even multidimensional delta-subharmonic spaces with any $1 \le q < +\infty$, both in the suitable analogs of the disc and the half-plane.

Chapter 2. Considering the complex parameters β (Re $\beta > 0$) and φ (Re $\varphi > -1$) instead of $\rho > 0$ and $\gamma > -1$ in [41], a some representations are found for the functions of the general spaces $H^p_{\alpha,\beta,\varphi}(\mathbb{D})$, $h^p_{\alpha,\beta,\varphi}(\mathbb{D})$ $(0 < p < +\infty\ \beta > 0,\ \alpha > -1,\ \varphi > -1)$ are investigated, which respectively are of functions $f(z)$ holomorphic, harmonic or just measurable in the unit disc $\mathbb{D}$ and satisfy the condition

$$\|f\|_{\alpha,\beta,\varphi} = \iint_{\mathbb{D}} |f(\zeta)|^p(1-|\zeta|^{2\beta})|\zeta|^{2\varphi}dm(\zeta) < +\infty,$$

where m is the Lebesgue surface measure. Note that for holomorphic and harmonic functions the above parameters β and φ are purely technical, since the classes $H^p_{\alpha,\beta,\varphi}(\mathbb{D})$ and $h^p_{\alpha,\beta,\varphi}(\mathbb{D})$ evidently coincide with H^p_α $(\equiv A^p_\alpha)$ and its harmonic version. Nevertheless β and φ participate in the above norm definition, since they appear in the found representations by surface integrals containing the Mittag-Leffler type reproducing kernels

$$S_{\alpha,\beta,\varphi}(z,\zeta) = \frac{\beta}{\pi\varphi(1+\alpha)}\sum_{k=0}^{\infty}\frac{\varphi(1+\mu+\alpha+k/\beta)}{\varphi(\mu+k/\beta)}z^k\bar{\zeta}^k, \quad \mu = \frac{1+\varphi}{\beta}.$$

The goal of the chapter is the passage to complex parameters, and the found representations differ from those of M.M.Djrbashian [41] mainly by the passage to complex β and φ. Note that a passage to complex parameters in the M.M.Djrbashian's factorization theory and related results was first suggested by Academician A.B.Nersesian long time ago.

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[1]Jerbashian, A.M., Pejendino, J.: Banach spaces of functions delta-subharmonic in the unit disc. In: Operator Theory and Harmonic Analysis, OTHA 2020, Part I New General Trends and Advances of the Theory, pp. 259-277. Springer (2021).

[2]Jerbashian, A.M., Vargas, D.F.: Banach spaces of functions delta-subharmonic in the half-plane. In: Operator Theory and Harmonic Analysis, OTHA 2020, Part I New General Trends and Advances of the Theory, pp. 279-296. Springer (2021).

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Chapter 3 is devoted to the proof of a far-reaching generalization of the Pompeu formula which then is used for finding a solution of the $\overline{\partial}$-equation. The following kernels are used:

$$S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi\Gamma(1+\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(1+\mu+\beta+k/\rho)}{\Gamma(\mu+k/\rho)}z^k\overline{\zeta}^k, \quad z,\zeta \in \mathbb{D},$$

and

$$Q_{\beta,\rho,\varphi}(z,\zeta) = 1+\frac{(z-\zeta)\rho}{\pi\Gamma(1+\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(1+\mu+\beta+k/\rho)}{\Gamma(\mu+k/\rho)}\frac{z^k}{\zeta^k}\int_0^{|\zeta|^2}(1-t^\rho)^\beta t^{\varphi+k}dt, \quad z\in\mathbb{D},$$

$\zeta \in \mathbb{D}\setminus\{0\}$, $Q_{\beta,\rho,\varphi}(z,0) \equiv 1$, and the below representation theorem is proved.

Theorem 0.12 *Let* $1 \le p < +\infty$, $\alpha > -1$, $\mathrm{Re}\,\varphi \ge \gamma > -1$, $\rho > 1$. *Also, let*

$$f \in L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \cup C^1(\mathbb{D}) \quad \textit{and} \quad \frac{\partial f(\zeta)}{\partial\overline{\zeta}} \in L^p_{1+\alpha,\rho,\gamma}(\mathbb{D}).$$

Then, for any $z \in \mathbb{D}$ *the function* $f(z)$ *is representable in the form*

$$f(z) = \iint_{\mathbb{D}} f(\zeta)S_{\beta,\rho,\varphi}(z,\zeta)(1-|\zeta|^{2\rho})^\beta|\zeta|^{2\varphi}dm(\zeta) - \frac{1}{\pi}\iint_{\mathbb{D}}\frac{\partial f(\zeta)/\partial\overline{\zeta}}{\zeta - z}Q_{\beta,\rho,\varphi}(z,\zeta)dm(\zeta).$$

Note that the last summand in the right-hand side of the above representation disapears for a holomorphic or harmonic function $f(z)$, while it is a solution of the $\overline{\partial}$-equation

$$\frac{\partial g(z)}{\partial\overline{z}} = v(z)$$

which plays an important role in complex analysis, especially in several complex variables, while even in case of one complex variable it had some important applications in the solution of the corona problem and in approximation theory. A solution of the $\overline{\partial}$-equation is found in the next theorem which is a generalization of a statement proved in [44, 45].

Theorem 0.13 $v(\zeta) \in C^k_0(\mathbb{D})$ *for some* $k \ge 1$, *i.e. is* k*-times continuously differentiable in* $\mathbb{D}$ *and ??, then the function*

$$g_{\beta,\rho,\varphi}(z) = -\frac{1}{\pi}\iint_{\mathbb{D}}\frac{v(\zeta)}{\zeta - z}Q_{\beta,\rho,\varphi}(z,\zeta)dm(\zeta), \quad z\in\mathbb{D},$$

where $\rho > 0$, $\mathrm{Re}\,\beta$, $\mathrm{Re}\,\varphi > -1$, $\mu = (\varphi+1)/\rho$, *is of the class* $C^k(\mathbb{D})$ *and is a solution of the* $\overline{\partial}$*-equation in* $\mathbb{D}$.

Chapter 4 is devoted to the proof of some half-plane analogs of the results of Chapter 3. To this end, on the base of the multidimensional representation formula in [51, Theorem2.2], the following operator is formally introduced:

$$T^*(u)(w) = -\frac{2^{\beta+1}}{\pi}\iint_{\prod_+}\frac{u(\eta)}{\eta - w}\frac{(\mathrm{Im}\,\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}}dm(\eta), \quad w \in \prod\nolimits_+,$$

where it is assumed that $\mathrm{Re}\,\beta > -1$ and $u(\eta)$ is a complex, measurable function in $\prod_+$. Using the Cayley transform, this operator is connected with the corresponding integral operator considered in [43], which gives some weighted solutions of the $\overline{\partial}$-equation in $\mathbb{D}$.

This results in various restrictions on the function $u(\eta)$, which provide that $T^*(u)(w)$ is a solution of the $\overline{\partial}$-equation in $\prod_+$. The main results are proved in the next two theorems.

Theorem 0.15 *Let a function $u \in C^k(\prod_+)$ $(k \geq 1)$ satisfy one of the conditions:*

(a) $u(\eta)\mathrm{Im}\,\eta \in L^1_{\alpha,\gamma}(\prod_+)$ *with* $\alpha > -1$, $\mathrm{Re}\,\beta \geq \alpha > -1$ *and* $\gamma \leq 2+\alpha$;

(b) $u(\eta)\mathrm{Im}\,\eta \in L^p_{\alpha,\gamma}(\prod_+)$ *with* $1 < p < +\infty$, $\alpha > -1$, $\mathrm{Re}\,\beta \geq (\alpha+1/p-1$ *and* $\gamma \leq 2+\alpha$.

Then, the function $f_\beta(w) = T^*_\eta(u)(w)$ is a solution of the $\overline{\partial}$-equation

$$\frac{\partial f(w)}{\partial \overline{w}} = u(w), \quad w \in \prod\nolimits_+.$$

Theorem 0.16 *Let $1 \leq p < +\infty$, $\alpha > -1$, $\gamma \in \mathbb{R}$. Besides, let*

$$4p + p\mathrm{Re}\,\beta + \gamma - 4 - 2\alpha \geq 0 \quad \text{and} \quad \mathrm{Re}\,\beta > 4p + p\mathrm{Re}\,\beta + \gamma - 5 - \alpha > -1.$$

Then, for any $u(w) \in C^k(\prod_+) \cup L^p_{\alpha,\gamma}(\prod_+)$ $(k \geq 1)$ the function $f_\beta(w) = T^*_\beta(u)(w)$ is a solution of the $\overline{\partial}$-equation in $\prod_+$. Moreover, the following estimate is true:

$$\|f_\beta\|_{p,4p+p\mathrm{Re}\,\beta+\gamma-5-\alpha} \leq \mathrm{const}(p,\beta,\alpha,\gamma)\|u\|_{p,\beta,\alpha,\gamma}.$$

The main results of the Thesis are published in the papers [73]-[79] of its author.

CONCLUSION

The Candidate Thesis "Weighted Integral Representations and Properties of Some Weighted Classes of Holomorphic, Harmonic and Smooth Functions in the Unit Disc and Half-Plane" of Feliks Hayrapetyan is a high level investigation work devoted to the solutions of some actual questions of Complex Analysis. All the results of the Thesis are new and undoubtedly are an essential contribution to an actual topics of Complex Analysis. The Sinopsis corresponds to the Thesis.

In my opinion, the Thesis satisfies all the requirements and its author Feliks Hayrapetyan deserves the degree of Candidate of Physical and Mathematical Sciences A.01.01-Mathematical Analysis.

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November 1, 2021