ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Նավասարդ Կարենի Վարդանյան

Պատիկ հատման կետերի և GC_n բազմությունների վերաբերյալ

Ա 01.07 «Հաշվողական մաթեմատիկա» մասնագիտությամբ
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ԵՐԵՎԱՆ – 2023

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On multiple intersection points and GC_n sets

SYNOPSIS

Of dissertation for requesting the degree of candidate of
physical and mathematical sciences specializing in
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YEREVAN - 2023

Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում

Գիտական ղեկավար՝ ֆիզ. մաթ. գիտ. դոկտոր Հ. Ա. Հակոբյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ. մաթ. գիտ. դոկտոր Մ. Գ. Գրիգորյան
ֆիզ. մաթ. գիտ. թեկնածու Դ. Ս. Ոսկանյան

Առաջատար կազմակերպություն՝ ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտ

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Մասնագիտական խորհրդի գիտական քարտուղար
ֆիզ. մաթ. գիտ. դոկտոր Տ. Ն. Հարությունյան

The topic of dissertation is approved in the Yerevan State University

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The dissertation defense will take place on January 31st, 2023, at 15:00, during the meeting of the Supreme Certifying Committee Specialized Council 050 "Computational Mathematics" at YSU (1 A. Manukyan, Yerevan 0025, Armenia).

The dissertation is available in the library of Yerevan State University.
The Synopsis was sent on December 20th, 2022.

Scientific secretary of specialized council
Doctor of phys. math. sciences T. N. Harutyunyan

Actuality of the subject. When solving some problems in approximation theory, computational mathematics, and numerical analysis, we often substitute functions by simpler ones, in particular by polynomials. One of the most prevalent such methods is polynomial interpolation.

Polynomial interpolation in one variable is one of the essential tools in applied mathematics and has a well-developed theory. The first fundamental results of univariate polynomial interpolation have been obtained by Lagrange and Newton.

In contrast to the univariate setting, multivariate polynomial interpolation is far from being developed. Even many fundamental problems of multivariate interpolation are still unsolved.

The dissertation deals with such a problem, namely the Conjecture of Gasca and Maeztu, which was put forward in 1982.

Essentially, the investigation of the multivariate interpolation problem began in the second half of the 20th century, especially due to the development of computers. Another reason for the interest in multivariate problems was the emergence of new mathematical theories, such as cubature formulae, and finite element methods.

Let us mention that in the multivariate case, the correctness of the interpolation problem depends not only on the number of interpolation nodes as in the univariate case but also on the geometric distribution of the nodes.

The first important results of bivariate polynomial interpolation were obtained by Berzolari, Radon, as well as by Chang and Yao.

It is worth mentioning that the mulltivariate polynomial interpolation is closely related to algebraic geometry. Indeed, the correctness of a multivariate interpolation problem is equivalent to the fact that the interpolation nodes do not lie in an algebraic curve of a certain degree.

Purpose and goals of the thesis. The thesis consists of five chapters. In the first chapter we study univariate and bivariate polynomial interpolation. The second chapter is concerned with constant coefficient partial differential equation(PDE) systems and generalized intersection multiplicities described by partial differential operators. In the third chapter we present the Noether theorem and the Cayley-Bacharach theorem with the generalized multiplicities. The fourth chapter contains investigation of the basic properties of GC_n sets. Finally in the fifth chapter the Gasca-Maeztu conjecture for the case $n = 6$ is considered.

The object of research. Intersection point, constant coefficient PDE system, bivariate polynomial interpolation, n-correct set, n-independent set, n-fundamental polynomial, PD multiplicity space, arithmetical multiplicity, the Gasca–Maeztu conjecture, GC_n set, maximal line, proper line, algebraic curve, maximal curve.

The methods of research. The methods of univariate and multivariate polynomial interpolations are used. Also some methods of linear algebra, algebraic geometry and constant coefficient PDE are used.

Scientific novelty. Let p and q be bivariate polynomials. We consider the concept of the multiplicity of intersection points of two plane algebraic curves $p, q = 0$, based on partial differential operators. We evaluate the exact number of maximal linearly independent differential conditions of degree k for all $k \geq 0$. On the other hand this gives the exact number of maximal linearly independent polynomial and polynomial-exponential solutions, of given degree k, for homogeneous PDE system $p(D)f = 0,\ q(D)f = 0$.

Then we prove the Noether theorem with the multiplicities described by PD operators. It is worth mentioning that despite the known analogue versions in this case the provided conditions are necessary and sufficient. We also prove the Cayley-Bacharach theorem with PD multiplicities. As far as we know this is the first generalization of this theorem for multiple intersections.

Next the simplest n-correct sets in the plane: GC_n sets are studied. An n-correct set $\mathcal{X}$ is called GC_n set if the n-fundamental polynomial of each node is a product of n linear factors. We say that a node uses a line if the line is a factor of the fundamental polynomial of this node. A line is called k-node line if it passes through exactly k nodes of $\mathcal{X}$. At most $n+1$ nodes can be collinear in any GC_n set and an $(n+1)$-node line is called a maximal line.

The Gasca-Maeztu conjecture (1982) states that every GC_n set possesses a maximal line. Until now the conjecture has been proved only for the cases $n \leq 5$.

Here, for a line ℓ we introduce and study the concept of ℓ-lowering of a node set $\mathcal{X}$ and define so called proper lines. We also prove some new properties of GC_n sets regarding the n-node lines and the subset of nodes that use a given line. Also refinments of several basic properties of GC_n sets are provided.

Finally, an important step is made in proving the Gasca-Maestu conjecture in the case $n = 6$.

Practical significance. The results obtained in the thesis are theoretical. They have also practical importance.

The following provisions are presented for the defence.

- The exact number of maximal linearly independent differential conditions of degree k for all $k \geq 0$ is evaluated for multiple intersection points.
- The analogue of the Noether theorem with PD multiplicities is proved.
- Some new properties of GC_n sets regarding n-node lines and the subset of nodes that use a given line are provided.
- Refinements of several basic properties of GC_n sets regarding the maximal lines and the used lines are proved.
- A step for proving the case $n = 6$ of the Gasca-Maeztu conjecture has been taken.

The approbation of obtained results. The results of the thesis were reported

- in the scientific seminars held in the department of Numerical Analysis of Faculty of Informatics and Applied Mathematics of Yerevan State University,
- in the International Conference on Mathematical Analysis and Differential Equations, 19-23 September, 2022, Tsaghkadzor, Armenia, Abstracts, p. 65 ([5*]).

Publications. The results of the thesis were published in 3 scientific articles and reported in an international conference which we bring at the end of the Synopsis. One article is accepted for publication.

The structure and the content of the thesis. The thesis consists of introduction, two parts each of which contains three chapters, summary and bibliography. The publications of the author are [1*] - [4*]. The paper [5*] is accepted for publication. The number of references is 41. The content of the thesis is 113 pages.

THE CONTENT OF THE THESIS

In **Chapter 1** we present univariate and multivariate interpolation and some basic known facts.

Denote by Π_n the space of bivariate polynomials of total degree $\leq n$, for which
$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$
The space of polynomials in two variables is denoted by Π.
Let $\mathcal{X} := \mathcal{X}_s = \{(x_1, y_1), \ldots, (x_s, y_s)\}$ be a set of s distinct nodes in the plane.

The problem of finding a polynomial $p \in \Pi_n$ satisfying the conditions

$$p(x_i, y_i) = c_i, \qquad i = 1, 2, \ldots s, \tag{1.2.1}$$

for a data $\bar{c} := \{c_1, \ldots, c_s\}$ is called *interpolation problem.*

Definition 1.2.1. A set of nodes $\mathcal{X}_s$ is called *n-correct* if for any data $\bar{c}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.2.1).

A necessary condition of n-correctness is: $\#\mathcal{X}_s = s = N$.

Denote by $p|_{\mathcal{X}}$ the restriction of p on $\mathcal{X}$.

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for $A \in \mathcal{X}$ if

$$p|_{\mathcal{X} \setminus \{A\}} = 0 \text{ and } p(A) = 1.$$

We denote an n-fundamental polynomial of $A \in \mathcal{X}$ by $p_A^\star = p_{A,\mathcal{X}}^\star$.

Definition 1.2.3. A set of nodes $\mathcal{X}$ is called *n-independent* if each node has n-fundamental polynomial. Otherwise, it is *n-dependent.* A set $\mathcal{X}$ is called *essentially n-dependent* if none of its nodes has n-fundamental polynomial.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $\#\mathcal{X}_s = s \leq N$.

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$. In particular, by ℓ we denote a linear polynomial from Π_1 and the line defined by the equation $\ell(x, y) = 0$.

The following proposition is well-known (see e.g. [12] Prop. 1.3):

Proposition 1.2.5. Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell q,$ where $q \in \Pi_{n-1}$.

This implies that at most $n+1$ nodes of an n-independent set can be collinear. An $(n+1)$-node line ℓ is called a *maximal line* (C. de Boor, [1]).

Set $d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k)$.

The following is a generalization of Proposition (1.2.5).

Proposition 1.2.6 ([18], Prop. 3.1)**.** Let q be an algebraic curve of degree $k \leq n$ with no multiple components. Then the following hold:

(i) any subset of q containing more than $d(n, k)$ nodes is n-dependent;

(ii) any subset $\mathcal{X}$ of q containing exactly $d(n,k)$ nodes is n-independent if and only if

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}} = 0 \implies p = qr, \text{ where } r \in \Pi_{n-k}.$$

Thus at most $d(n,k)$ n-independent nodes lie in a curve q of degree $k \le n$.

Definition 1.2.7. Let $\mathcal{X}$ be an n-independent set of nodes with $\#\mathcal{X} \ge d(n,k)$. A curve of degree $k \le n$ passing through $d(n,k)$ points of $\mathcal{X}$ is called maximal.

The following is a characterization of the maximal curves:

Proposition 1.2.8 ([18], Prop. 3.3)**.** Let $\mathcal{X}$ be an n-independent set of nodes with with $\#\mathcal{X} \ge d(n,k)$. Then a curve μ of degree k, $k \le n$, is a maximal curve if and only if

$$p \in \Pi_n,\ p|_{\mathcal{X}\cap\mu} = 0 \implies p = \mu s,\ s \in \Pi_{n-k}.$$

Next consider special n-correct sets: GC_n sets, defined by Chung and Yao:

Definition 1.2.13 ([8])**.** An n-correct set $\mathcal{X}$ is called GC_n *set*, if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

Now we are in a position to present the Gasca-Maeztu, or briefly GM

Conjecture 1.2.14 ([9], 1982)**.** Any GC_n set contains $n+1$ collinear nodes.

So far, the GM conjecture has been confirmed to be true only for $n \le 5$. The case $n = 2$ is trivial. The case $n = 3$ was established by M. Gasca and J. I. Maeztu in [9]. The case $n = 4$ was proved by J. R. Busch [2]. Other proofs of this case have been published since then (see e.g. [4], [12]). The case $n = 5$ was proved by H. Hakopian, K. Jetter and G. Zimmermann [13]. Recently G. Vardanyan provided a simpler and shorter proof for this case [19].

Definition 1.2.15. Let $\mathcal{X}$ be an n-correct set. We say, that a node $A \in \mathcal{X}$ *uses a line* ℓ, if $p_A^\star = \ell q, \quad q \in \Pi_{n-1}$.

Definition 1.2.17. For a given set of lines $\ell_1, \ldots, \ell_k$, we define $\mathcal{N}_{\ell_1,\ldots,\ell_k}$ to be the set of those nodes in $\mathcal{X}$ which do not lie in any ℓ_i, and for which at least one of the lines is not used.

In the case of one line ℓ we have

$$\mathcal{N}_\ell = \{A \in \mathcal{X} : A \notin \ell, \text{ and } A \text{ is not using } \ell\}.$$

Proposition 1.2.18 ([12], Thm. 3.2)**.** Assume that $\mathcal{X}$ is a GC_n set, and $\ell_1, \ldots, \ell_k$ are lines. Then the following hold for $\mathcal{N} = \mathcal{N}_{\ell_1,\ldots,\ell_k}$.

(i) If $\mathcal{N}$ is nonempty, then it is essentially $(n-k)$-dependent.

(ii) $\mathcal{N} = \emptyset$ if and only if the product $\ell_1 \cdots \ell_k$ is a maximal curve.

In **Chapter 2** we present constant coefficient PDE systems and intersection multiplicities.

The two variables are denoted by $\mathbf{x} = (x_1, x_2)$ or sometimes by (x, y). For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ set $\alpha! = \alpha_1!\ \alpha_2!,\quad |\alpha| = \alpha_1 + \alpha_2$. For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, denote:

$$\mathbf{xy} = x_1y_1 + x_2y_2,\quad \mathbf{x}^\alpha := {x_1}^{\alpha_1}{x_2}^{\alpha_2}.$$

The differential operator given by the polynomial $r \in \Pi$ is denoted by

$$r(D) := r(D_1, D_2),\ r^{(\alpha)} := D^\alpha r := (D_1)^{\alpha_1}(D_2)^{\alpha_2} r,\ \text{where } D_i := D_{x_i}.$$

Next we bring the *PD multiplicity space* ([17], [10], [16]) for $\lambda \in r \in \Pi$:

$$\mathcal{M}_\lambda(r) = \left\{h \in \Pi : D^\alpha h(D) r(\lambda) = 0\ \forall \alpha \in \mathbb{Z}_+^2\right\}.$$

Denote by $\mathcal{Z}_0 = p \cap q$ the set of intersection points of curves $p, q \in \Pi$.

Definition 2.1.1. Suppose that $p, q \in \Pi$ and $\lambda \in \mathcal{Z}_0$. Then the following space is called the multiplicity space of the intersection point λ :

$$\mathcal{M}_\lambda(p, q) = \mathcal{M}_\lambda(p) \cap \mathcal{M}_\lambda(q).$$

The number $\mu_\lambda(p, q) := \dim \mathcal{M}_\lambda(p, q)$ is called the arithmetical multiplicity of the point λ.

Let us start with the following result (see also [15], Theorem 6):

Theorem 2.1.3 ([10], Theorem 5)**.** Suppose that λ is a solution of an algebraic equation

$$r(\mathbf{x}) = 0, \quad r \in \Pi.$$

Then a polynomial $h \in \Pi$ belongs to the multiplicity space $\mathcal{M}_\lambda(r)$ if and only if the function

$$y = h(\mathbf{x}) exp(\lambda \mathbf{x})$$

is a solution of the PDE

$$r(D) y = 0. \tag{2.1.3}$$

In particular for $\lambda = \theta := (0, 0)$ the following relation holds

$$h \in \mathcal{M}_\theta(r) \iff r(D)h = 0, \text{where } r, h \in \Pi.$$

Denote the space of polynomial-exponential solutions of PDE (2.1.3) by

$$\mathcal{S}_\lambda(r) := \{y = h(\mathbf{x}) exp(\lambda \mathbf{x}) : r(D) y = 0,\ h \in \Pi\}.$$

For $p, q \in \Pi$, consider the following PDE system:

$$\begin{cases} p(D) f = 0, \\ q(D) f = 0. \end{cases} \tag{2.1.4}$$

The corresponding space of solutions for PDE system denote by

$$\mathcal{S}_\lambda(p, q) := \mathcal{S}_\lambda(p) \cap \mathcal{S}_\lambda(q).$$

Denote for $\lambda \in r \in \Pi$:

$$\mathcal{M}_{k,\lambda}(r) := \mathcal{M}_{\lambda}(r) \cap \Pi_k, \quad \mathcal{S}_{k,\lambda}(r) := \mathcal{S}_{\lambda}(r) \cap \Pi_k.$$

In view of Theorem 2.1.3 the following equality holds

$$\dim \mathcal{M}_{k,\lambda}(r) = \dim \mathcal{S}_{k,\lambda}(r), \quad r \in \Pi. \tag{2.2.1}$$

We say that λ is an m_0-fold zero for p if the least nonzero homogenous part of $p(\mathbf{x} + \lambda)$ is the m_0-homogeneous part.

Theorem 2.2.1 ([1*])**.** Suppose that $p \in \Pi_m$ is a polynomial for which λ is an m_0-fold zero. Then the PD equation $p(D)f = 0$ has exactly D_k linearly independent solutions of the form

$$h(\mathbf{x})exp(\lambda\mathbf{x}), \quad h \in \Pi_k,$$

where D_k is the kth partial sum of the following series:

$$\sum_{i=0}^{\infty} d_i := 1 + 2 + \cdots + m_0 + m_0 + \cdots + m_0 + \cdots . \tag{2.2.2}$$

In view of (2.2.1) we obtain

Corollary 2.2.2 ([1*])**.** Suppose that $p \in \Pi$ is a polynomial for which λ is an m_0-fold zero. Then there are exactly D_k linearly independent polynomials in the space $\mathcal{M}_{k,\lambda}(p)$, where D_k is the kth partial sum of the series (2.2.2).

For the next result we accept a very common restriction from the theory of intersection. Namely, we assume that the two polynomials p and q have no common tangent line at an intersection point $\lambda \in \mathcal{Z}_0$. This means that the lowest homogeneous parts of the polynomials $p(\mathbf{x} + \lambda)$ and $q(\mathbf{x} + \lambda)$, have no common factor.

Theorem 2.2.3 ([1*])**.** Suppose that polynomials $p, q \in \Pi$, have no common tangent line at an intersection point $\lambda \in \mathcal{Z}_0$. Suppose also that for p and q the point λ is an m_0 and n_0-fold zero, respectively, $m_0 \le n_0$. Then the PDE system (2.1.4) has exactly D_k linearly independent solutions of the form

$$h(\mathbf{x})exp(\lambda\mathbf{x}), \quad h \in \Pi_k,$$

where D_k is the kth partial sum of the following series:

$$\sum_{i=0}^{\infty} d_i := 1 + 2 + \cdots + (m_0 - 1) + \underbrace{m_0 + \cdots + m_0}_{n_0 - m_0 + 1} + (m_0 - 1) + \cdots + 1 + 0 \cdots + 0 + \cdots . \tag{2.2.3}$$

Corollary 2.2.4 ([1*])**.** Suppose that $p \in \Pi$ is a polynomial for which λ is an m_0-fold zero. Then there are exactly D_k linearly independent polynomials in the space $\mathcal{M}_{k,\lambda}(p)$where D_k is the kth partial sum of the series (2.2.3).

In **Chapter 3** the Noether and the Cayley-Bacharach theorems with PD multiplicities are presented.

Consider a set of s linear operators (functionals) on Π_n :

$$\mathcal{L}_s = \{L_1, \ldots, L_s\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$L_i p = c_i, \qquad i = 1, 2, \ldots s, \tag{3.1.2}$$

is called the Lagrange interpolation problem with operators.

We consider linear operators L which are partial differential operators evaluated at points:

$$Lf = p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f\big|_{(x_0,y_0)},$$

where $p \in \Pi$. We say that L has degree d, where $d = \deg p$.

Definition 3.1.1. A set of operators $\mathcal{L}_s$ is called *n-correct* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (3.1.2).

A necessary condition of n-correctness of $\mathcal{L}_s$ is: $|\mathcal{L}_s| = s = N$.

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for an operator $L_k \in \mathcal{X}_s$ if

$$L_i p = \delta_{ik},\ \ i = 1, \ldots, s,$$

where δ is the Kronecker symbol.

We denote the n-fundamental polynomial for $L \in \mathcal{L}_s$ by $p_L^\star = p_{L,\mathcal{L}}^\star$. Sometimes we also call fundamental a polynomial at which vanish all operators but one, since it is a nonzero constant times the fundamental polynomial.

Next we introduce an important concept of n-dependence of sets of operators:

Definition 3.1.3. A set of operators $\mathcal{L}$ is called *n-independent* if each operator has a fundamental polynomial in Π_n. Otherwise, $\mathcal{L}$ is called *n-dependent.*

Clearly fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence of the set $\mathcal{L}$ is $|\mathcal{L}| \leq N$.

Suppose λ is a point in the plane. Consider the operator L_λ defined by $L_\lambda f = f(\lambda)$. We say that a set of points $\mathcal{X}$ is n-independent (n-correct) if the set of operators $\{L_\lambda : \lambda \in \mathcal{X}\}$ is n-independent (n-correct).

Suppose a set of operators $\mathcal{L}$ is n-independent. Then by using the Lagrange formula:

$$p = \sum_{L \in \mathcal{L}} c_L p_{L,\mathcal{L}}^\star, \quad c_L = Lp,$$

we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (3.1.2).

Let ℓ be a line. We say that $p \in \Pi$ vanishes at $\lambda \in \ell$ with the multiplicity m if

$$(D_a)^i p\big|_\lambda = 0, \ i = 0, \dots, m-1,$$

where $a||\ell$ and D_a is the directional derivative.

The following proposition is well-known (see, e.g., [12] Proposition 1.3):

Proposition 3.1.4. Suppose that ℓ is a line and a polynomial $p \in \Pi_n$ vanishes at some points of ℓ with the sum of multiplicities $n+1$. Then we have

$$p = \ell r, \ \text{where } r \in \Pi_{n-1}. \tag{3.1.3}$$

Next, we present the Noether theorem with the multiplicities described by PD operators.

Theorem 3.2.2 ([2*])**.** Suppose that polynomials $p, q \in \Pi, \ \deg p = m, \ \deg q = n$, have no intersection point at infinity. Suppose also that $f \in \Pi_k$ vanishes at $\mathcal{M}_\lambda(p,q)$ for each $\lambda \in p \cap q$. Then we have that

$$f = Ap + Bq, \tag{3.2.1}$$

where $A \in \Pi_{k-m}, \ B \in \Pi_{k-n}$.

Note that the inverse theorem is true. Indeed, if (4.6.2) holds then $f \in \Pi_k$ and, in view of the formula (3.1.4), we have that and f vanishes at $\mathcal{M}_\lambda(p,q)$ for each $\lambda \in p \cap q$.

At the end let us bring the formulation of Theorem 3.2.2 in the homogeneous case.

Theorem 3.2.3 ([2*])**.** Suppose that $p \in \Pi_m^0$ and $q \in \Pi_n^0$ have no common component. Suppose also that $f \in \Pi_k^0$ vanishes at $\mathcal{M}_\lambda^0(p,q)$ for each $\lambda \in p \cap q$. Then we have that

$$f = Ap + Bq,$$

where $A \in \Pi_{k-m}^0, \ B \in \Pi_{k-n}^0$.

It is known that the set $\mathcal{Z}_0 := p \cap q$, where p and q are polynomials, of degree m and n, respectively, is $(m+n-2)$-independent, provided that $|\mathcal{Z}_0| = mn$. Below we prove this result without the last restriction (cf. [11], Corollary 1).

Corollary 3.2.4 ([2*])**.** Suppose that polynomials $p, q \in \Pi, \ \deg p = m, \ \deg q = n$, have no common component. Then the set of linear operators $\mathcal{L}(p,q)$ and consequently the set $\mathcal{Z}_0$ are $(m+n-2)$-independent.

Theorem 3.3.1 ([2*])**.** Suppose that polynomials $p, q \in \Pi, \ \deg p = m, \ \deg q = n$, have no intersection point at infinity and $\lambda \in \mathcal{Z}_0$. Suppose also that p and q have no common tangent line at λ. Then we have that the set of linear operators $\mathcal{L}^\lambda(p,q)$ contains only one operator of the highest degree: $\bar{L}$. Suppose also that $f \in \Pi_{m+n-3}$ vanishes at $\mathcal{L}(p,q) \setminus \{\bar{L}\}$. Then we have that f vanishes at all $\mathcal{L}(p,q)$.

In **Chapter 4** we present the basic properties of GC_n sets.

Denote by $M(\mathcal{X})$ the set of maximal lines of the node set $\mathcal{X}$.

The concept of the defect introduced by Carnicer and Gasca in [3] is an important characteristic of GC_n sets: $\text{def}(\mathcal{X}) := n + 2 - \#M(\mathcal{X})$. We have that $0 \le \text{def}(\mathcal{X}) \le n+2$.

Proposition 4.1.10 ([6], Crl. 3.5)**.** Let λ be a maximal line of a GC_n set $\mathcal{X}$ such that $\#M(\mathcal{X}\setminus\lambda) \geq 3$. Then we have that

$$\operatorname{def}(\mathcal{X} \setminus \lambda) = \operatorname{def}(\mathcal{X}) \quad \text{or} \quad \operatorname{def}(\mathcal{X}) - 1.$$

Definition 4.1.12 ([4])**.** Given an n-correct set $\mathcal{X}$ and a line ℓ, $\mathcal{X}^\ell$ is the subset of nodes of $\mathcal{X}$ which use the line ℓ.

Let $\mathcal{X}$ be an n-correct set, and ℓ be a line. Then
(i) a maximal line λ is called *ℓ-disjoint* if $\lambda \cap \ell \cap \mathcal{X} = \emptyset$;
(ii) two maximal lines λ', λ'' are called *ℓ-adjoint* if $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$.
The following two lemmas of Carnicer and Gasca play an important role in the sequel.

Lemma 4.1.13 ([6], Lemma 4.4)**.** Let $\mathcal{X}$ be an n-correct set and ℓ be a line with $\#(\ell \cap \mathcal{X}) \leq n$. Suppose also that a maximal line λ is ℓ-disjoint. Then we have that $\mathcal{X}^\ell = (\mathcal{X} \setminus \lambda)^\ell$.

The set $\mathcal{X} \setminus \lambda$ is called *ℓ-disjoint reduction* of $\mathcal{X}$.

Lemma 4.1.14 ([6], Proof of Thm. 4.5)**.** Let $\mathcal{X}$ be an n-correct set and ℓ be a line with $\#(\ell\cap\mathcal{X}) \leq n$. Suppose also that two maximal lines λ', λ'' are ℓ-adjoint. Then we have that $\mathcal{X}^\ell = [\mathcal{X} \setminus (\lambda' \cup \lambda'')]^\ell$.

The set $\mathcal{X} \setminus (\lambda' \cup \lambda'')$ is called *ℓ-adjoint reduction* of $\mathcal{X}$.
Next, by the motivation of the above two lemmas, we introduce the concept of *ℓ-lowering* of a GC_n set.

Definition 4.1.15. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $2 \leq k \leq n+1$. We say that the set $\hat{\mathcal{X}} = \hat{\mathcal{X}}(\ell)$ is the ℓ-lowering of $\mathcal{X}$, and briefly denote this by $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$, if

$$\hat{\mathcal{X}} = \mathcal{X} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)\,,$$

where $\mathcal{U}_1$ is the union of the ℓ-disjoint maximal lines of $\mathcal{X}$, and $\mathcal{U}_2$ is the union of the (pairs of) ℓ-adjoint maximal lines of $\mathcal{X}$.

A maximal line ℓ has no ℓ-disjoint or ℓ-adjoint maximal lines. Thus, if ℓ is a maximal line then $\mathcal{U}_1 = \mathcal{U}_2 = \emptyset$ and hence $\hat{\mathcal{X}} = \mathcal{X}$.
From Lemmas (4.1.13) and (4.1.14) we immediately get that

$$\mathcal{X} \downarrow_\ell \hat{\mathcal{X}} \Rightarrow \mathcal{X}^\ell = \hat{\mathcal{X}}^\ell.$$

Definition 4.1.16. A node $A \in \mathcal{X}$ is called *k_m-node* if it belongs to exactly k maximal lines.

There are 0_m, 1_m, and 2_m-nodes only. It is easily seen that $\ell \cap \hat{\mathcal{X}}(\ell)$ contains no 2_m-node of $\mathcal{X}$.

Definition 4.1.18. Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node line, $2 \leq k \leq n$ and $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$. Then the line ℓ is called *proper* if it is a maximal line in the set $\hat{\mathcal{X}}$.
The line ℓ is called *proper* $(-r)$ if it becomes a maximal line after r steps of application of ℓ-disjoint or ℓ-adjoint reductions, in all, starting with $\hat{\mathcal{X}}$.
We call the ℓ-disjoint maximal line, or the union of the pair of ℓ-adjoint maximal lines, used at the above-mentioned kth step, the $(-k)$ *d/a (disjoint/adjoint)* item, $k = 1, \ldots, r$.

Denote by $Pr(\mathcal{X})$ the set of proper lines of $\mathcal{X}$. Let us present here the following

Theorem 4.1.19 ([7])**.** Let $\mathcal{X}$ be a GC_n set. Assume that the GM Conjecture holds for all degrees up to n. Then $\operatorname{def}(\mathcal{X}) \in \{0, 1, 2, 3, n-1\}$.

For GC_n sets of defect $n-1$ there is a simple formula for the set $\mathcal{X}^\ell$. For other GC_n sets we have the following

Theorem 4.6.1 ([3*])**.** Let $\mathcal{X}$ be a GC_n set, $\ell \notin M(\mathcal{X})$ be a used line and $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$. Assume that GM Conjecture holds for all degrees up to n. Then we have that

$$\operatorname{def}(\hat{\mathcal{X}}) = \operatorname{def}(\mathcal{X}) - 1 \quad \text{or} \quad \operatorname{def}(\mathcal{X}) - 2. \tag{4.6.1}$$

Next, by assuming that $\operatorname{def}(\mathcal{X}) \neq n-1$ if $n \geq 5$, and the line ℓ is not proper, we get that ℓ is proper $(-r)$ and has at most r $\hat{2}_m$-nodes, where $r = 1$ or 2. Also we have that the set $\mathcal{X}^\ell$ is GC_0 or GC_1 set and the set $\hat{\mathcal{X}}$ is a GC_k set with $k \leq 5$.
Moreover, if $S \in \ell$ is a $\hat{2}_m$-node, then $S = \ell^o \cap \lambda,$ where $\ell^o \in Pr(\mathcal{X}) \cap M(\hat{\mathcal{X}})$ and $\lambda \in M(\mathcal{X})$.

Next let us mention a result from [14] (Corollary 4.1), which states that if $n \geq 6$ then for any k-node line in a GC_n set $\mathcal{X},\ 2 \leq k \leq n$, there is either ℓ-disjoint maximal line or a pair of ℓ-adjoint maximal lines. Below we strengthen this result in the case of used lines.

Corollary 4.6.3 ([3*])**.** Let $\mathcal{X}$ be a GC_n set, ℓ be a k-node used line, $2 \leq k \leq n$. Assume that GM Conjecture holds for all degrees up to n. Then there is either ℓ-disjoint maximal line or a pair of ℓ-adjoint maximal lines.

Theorem 4.6.4 ([3*])**.** Let $\mathcal{X}$ be a GC_n set and ℓ be a k-node used line. Assume that ℓ contains exactly r 2_m-nodes and $\hat{r}$ $\hat{2}_m$-nodes. Assume that GM Conjecture holds for all degrees up to n. Then $\mathcal{X}^\ell$ is a GC_{s-2} set and hence $\#\mathcal{X}^\ell = \binom{s}{2}$, where $s = k - r - \hat{r}$. Moreover, for any used line ℓ we have that $\hat{r} \leq 2$. Furthermore, $\hat{r} = 0$ if $\#\mathcal{X}^\ell > 3$.

Below, we restate a result from [14] (Theorem 3.1). In the "Moreover" and "Furthermore" parts we complement it.
Let us call the maximal line of $\mathcal{X}$ passing through an $\hat{2}_m$-node in ℓ : *ℓ-special maximal.*

Theorem 4.6.6 ([3*])**.** Let $\mathcal{X}$ be a GC_n set, ℓ be a line with $\#\mathcal{X}^\ell = \binom{s}{2},\ s \geq 2$. Assume that GM Conjecture holds for all degrees up to n. Then for any maximal line λ we have that $\#(\lambda \cap \mathcal{X}^\ell) = s-1$, or 0.
Moreover, the latter case:

$$\#(\lambda \cap \mathcal{X}^\ell) = 0, \tag{4.6.2}$$

holds, if and only if either λ is an ℓ-disjoint maximal line or λ is one of ℓ-adjoint maximal lines or λ is an ℓ-special maximal line.

Recall that $N(\mathcal{X})$ denotes the set of all n-node lines of GC_n set $\mathcal{X}$. The following Proposition presents some properties of the set $N(\mathcal{X})$.

Proposition 4.7.1 ([3*])**.** Let $\mathcal{X}$ be a GC_n set, $n \geq 4$. Assume that GM Conjecture holds for all degrees up to n. Then $N(\mathcal{X}) \subset Pr(\mathcal{X})$ and $\#N(\mathcal{X}) \in \{0, 1, 2, 3\}$. Moreover, any n-node line intersects each maximal line, except possibly one, at a node of $\mathcal{X}$. Furthermore, any two n-node lines intersect at a node of $\mathcal{X}$.

Recall that in defect 1 sets the proper lines are the lines passing through at least two of $n+1$ 1_m-nodes. For other GC sets we have the following

Proposition 4.7.3 ([3*])**.** Let $\mathcal{X}$ be a GC_n set, $\mathrm{def}(\mathcal{X}) \neq 1,\ n \geq 4$. Assume that GM Conjecture holds for all degrees up to n. Then $\#Pr(\mathcal{X}) \in \{0, 3\}$.

For a defect $n-1$ set there is a simple formula for the fundamental polynomials. This formula gives the n used lines of the node A. For other GC_n sets we have the following

Proposition 4.7.5 ([3*])**.** Let $\mathcal{X}$ be a GC_n set and $\mathrm{def}(\mathcal{X}) \neq n-1$ if $n \geq 5$. Then we have that among n lines used by a node $A \in \mathcal{X}$ at most three lines are proper, at most one line is proper (-1), at most one line is proper (-2), and at least $n-3$ lines are maximal.

In **Chapter 5** we make a step for proving the Gasca-Maeztu conjecture for $n = 6$.

Suppose that $\mathcal{X}$ is a GC_n set. Consider a node $A \in \mathcal{X}$ together with the set of n used lines denoted by $\mathcal{L}_A$. The $N-1$ nodes of $\mathcal{X} \setminus \{A\}$ belong to the lines of $\mathcal{L}_A$. Let us order the lines of $\mathcal{L}_A$ in the following way:

The line ℓ_1 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X}$, denoted by k_1 : $\mathcal{X} \cap \ell_1 = k_1$. The line ℓ_2 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X} \setminus \ell_1$, denoted by $k_2 : (\mathcal{X} \setminus \ell_1) \cap \ell_2 = k_2$.

In the general case the line $\ell_s,\ s = 1, \ldots, n$, is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of the set $\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i$, denoted by $k_s : (\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i) \cap \ell_s = k_s$. A correspondingly ordered line sequence

$$\mathcal{S} = (\ell_1, \ldots, \ell_n)$$

is called a *maximal line sequence* or briefly an *m-line sequence* if the respective sequence $(k_1, \ldots, k_n)$ is the maximal in the lexicographic order [13]. Then the latter sequence is called a *maximal distribution sequence* or briefly an *m-d sequence.* Evidently, for the m-d sequence we have that

$$k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_1 + \cdots + k_n = N - 1. \tag{5.1.1}$$

An intersection point of several lines of $\mathcal{L}_A$ is counted for the line containing it which appears in $\mathcal{S}$ first. A node in $\mathcal{X}$ is called *primary* for the line it is counted for, and *secondary* for the other lines containing it.

In some cases a particular line $\widetilde{\ell}$ used by a node is fixed and then the properties of the other factors of the fundamental polynomial are studied.

In this case in the corresponding m-line sequence, called *$\widetilde{\ell}$-m-line sequence*, one takes as the first line ℓ_1 the line $\widetilde{\ell}$, no matter through how many nodes it passes. Then the second and subsequent lines are chosen, as in the case of the m-line sequence. Thus the line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\widetilde{\ell}_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \widetilde{\ell}_1$, and so on.

Correspondingly the *$\widetilde{\ell}$-m-distribution sequence* is defined.

Now let us formulate the Gasca-Maeztu conjecture for $n = 6$ as:

Theorem 5.2.1. Any GC_6 set contains seven collinear nodes.

To make a step for the proof assume by way of contradiction:

Assumption. *The set $\mathcal{X}$ is a GC_6 set without a maximal line.*

The only possible m-distribution sequences for any node $A \in \mathcal{X}$ in the case $n = 6$ with $N = 28$ are

$$\begin{array}{llllll} \text{(i)} & (6,6,6,4,3,2) & \text{(ii)} & (6,6,5,5,3,2) & \text{(iii)} & (6,6,5,4,4,2) \\ \text{(iv)} & (6,6,5,4,3,3) & \text{(v)} & (6,6,4,4,4,3) & \text{(vi)} & (6,5,5,5,4,2) \\ \text{(vii)} & (6,5,5,5,3,3) & \text{(viii)} & (6,5,5,4,4,3) & \text{(ix)} & (6,5,4,4,4,4) \\ \text{(x)} & (5,5,5,5,5,2) & \text{(xi)} & (5,5,5,5,4,3) & \text{(xii)} & (5,5,5,4,4,4). \end{array}$$

Consider a 2-node line $\widetilde{\ell}$. Note that in $\widetilde{\ell}$-m-d sequences, we use the tilde to indicate the place of $\widetilde{\ell}$. It was proved in [5], Prop. 4.2, that any 2-node line in a GC_n set $\mathcal{X}$ can be used at most by one node from $\mathcal{X}$. This yields the following

Proposition 5.3.1. Assume that $\mathcal{X}$ is a GC_6-set, and suppose that $\widetilde{\ell}$ is a 2-node line. Then $\widetilde{\ell}$ can be used by at most one node $A \in \mathcal{X}$.

Then, consider a 3-node line $\widetilde{\ell}$. Here, and in all subsequent cases, denote a respective $\widetilde{\ell}$-m-line sequence by $(\widetilde{\ell}, \ell_2, \ldots, \ell_6)$. We have shown the following:

Proposition 5.3.2 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a 3-node line $\widetilde{\ell}$ is used by two nodes A, $B \in \mathcal{X}$. Then there exists a third node C using $\widetilde{\ell}$ and $\widetilde{\ell}$ is used by exactly three nodes of $\mathcal{X}$.

Moreover, A, B, and C, share four lines with either $6,6,6,4$, or $6,6,5,5$, primary nodes, respectively.

Now, consider a 4-node line $\widetilde{\ell}$. We obtain the following

Proposition 5.3.3 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a 4-node line $\widetilde{\ell}$ is used by three nodes A, B, $C \in \mathcal{X}$. Then, A, B, and C, besides $\widetilde{\ell}$, share four lines with either $6,6,6,3$; $6,6,5,4$; or $6,5,5,5$, primary nodes, respectively.

Proposition 5.3.4 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that some 4-node line $\widetilde{\ell}$ is used by four nodes A, B, C, $D \in \mathcal{X}$. Then, $\widetilde{\ell}$ is used by exactly 6 nodes.

Moreover, besides $\widetilde{\ell}$, these six nodes share also three other lines each passing through 6 primary nodes.

Now suppose that $\widetilde{\ell}$ is a 5-node line.

Proposition 5.3.6 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and $\widetilde{\ell}$ is a 5-node line used by five nodes of $\mathcal{X}$. Then it is used by exactly six nodes.

Moreover, besides $\widetilde{\ell}$, these six nodes share also three other lines passing through $6,6,5$ primary nodes, respectively.

Finally suppose that $\widetilde{\ell}$ is a 6-node line.

Proposition 5.3.7 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}$ is a 6-node line. Assume also that $\widetilde{\ell}$ is used by eight nodes of $\mathcal{X}$. Then it is used by exactly ten nodes of $\mathcal{X}$.

Moreover, these ten nodes form a GC_3 set and share two more lines with six primary nodes each.

Proposition 5.3.8 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}_i,\ i = 1, 2$, are two disjoint 6-node lines. Assume also that six nodes of $\mathcal{X}$are using $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$. Then, the six nodes besides $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ share either one more line with 6 primary nodes or two more lines each with 5 primary nodes. In the first case the lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are used by exactly ten nodes of $\mathcal{X}$ and in the second case they are used by exactly six nodes of $\mathcal{X}$.

Moreover, in the first and second cases the ten and six nodes form a GC_3 and GC_2 sets, respectively.

By using the above mentioned results, as well as, several other results we prove the following basic result of this chapter:

Proposition 5.3.9 ([5*])**.** Assume that $\mathcal{X}$ is a GC_6 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is $(6, 6, 6, 4, 3, 2)$.

List of Publications of the Author

[1*] N. Vardanyan, On constant coefficient PDE systems and intersection multiplicites, *Proc. YSU. Phys. and Math. Sci.*, **54**, 108–114 (2020)

[2*] H. Hakopian and N. Vardanyan, On the Noether and the Cayley-Bacharach theorems with PD multiplicities, *J. Cont. Math. Anal.*, **55**, 30-42 (2020)

[3*] H. Hakopian and N. Vardanyan, On the basic properties of GC_n sets, *Journal of Knot Theory and Its Ramifications*, **29**, 1-26 (2020)

[4*] H. Hakopian, G. Vardanyan, and N. Vardanyan, On the Gasca-Maeztu conjecture for n = 6, the International Conference on Mathematical Analysis and Differential Equations, 19-23 September, 2022, Tsaghkadzor, Armenia, Abstracts, p. 65.

[5*] H. Hakopian, G. Vardanyan, and N. Vardanyan, On the Gasca-Maeztu conjecture for n = 6, accepted in *J. Cont. Math. Anal.*

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Ամփոփում

Աշխատանքում ուսումնասիրվում է երկու հանրահաշվական կորերի հատման կետի պատիկության տարածությունը, հիմնված մասնակի ածանցյալներով դիֆերենցիալ օպերատորների վրա:

Դիցուք $p = 0$ և $q = 0$ հանրահաշվական կորեր են որոնց համար (x_0, y_0) հատման կետը համապատասխանաբար m_0 և n_0 կարգի զրո է: Ենթադրում ենք, որ կորերը չունեն ընդհանուր շոշափող: Երկրորդ գլխում մենք ստացել ենք պատիկ հատման կետի համար k աստիճանի մաքսիմալ գծորեն անկախ դիֆերենցիալ պայմանների ճշգրիտ թիվը յուրաքանչյուր ոչ բացասական k թվի համար: Մասնավորապես ապացուցվել է, որ դիֆերենցիալ պայմանների ամենամեծ աստիճանը m_0+n_0-2 է, և այս դեպքում կա ճիշտ մեկ գծորեն անկախ պայման:

Մյուս կողմից այստեղից ստանում ենք $p(D)f = 0$, $q(D)f = 0$ մասնակի ածանցյալներով դիֆերենցիալ հավասարումների համասեռ համակարգի k աստիճանի մաքսիմալ գծորեն անկախ բազմանդամային-էքսպոնենցիալ լուծումների ճշգրիտ քանակը:

Աշխատանքի երրորդ գլխում ապացուցվել է Նյոթերի թեորեմը, որտեղ պատիկությունները նկարագրվում են մասնակի ածանցյալներով դիֆերենցիալ օպերատորներով: Նշենք, որ ի տարբերություն Նյոթերի թեորեմի պատիկ դեպքի հայտնի տարբերակների, այստեղ բերված պայմանները անհրաժեշտ և բավարար են:

Այնուհետև ապացուցվել ենք Քելի-Բախարախի թեորեմը, որտեղ պատիկությունները նկարագրվում են մասնակի ածանցյալներով դիֆերենցիալ օպերատորներով: Ինչքան մեզ հայտնի է սա Քելի-Բախարախի թեորեմի առաջին ընդհանրացումն է պատիկ հատման կետերի դեպքի համար:

Չորրորդ գլխում ուսումնասիրվում են հարթության մեջ պարզագույն n-ճշգրիտ բազմությունները՝ GC_n բազմությունները: Հանգույցների n-ճշգրիտ բազմությունը անվանում ենք GC_n բազմություն, եթե յուրաքանչյուր հանգույցի ֆունդամենտալ բազմանդամ գծային արտադրիչների արտադրյալ է: Կասենք, որ կետը օգտագործում է տրված l ուղիղը, եթե l-ը այդ կետի ֆունդամենտալ բազմանդամի արտադրիչ է: Ուղիղը կանվանենք k-հանգույցանի, եթե այն անցնում է X բազմության ճիշտ k հանգույցներով: Հանգույցների n-ճշգրիտ բազմության մեջ ամենաշատը $n+1$ հանգույց կարող են գտնվել մեկ ուղղի վրա և $(n+1)$-հանգույցանի ուղիղներն անվանում են մաքսիմալ:

Տրված l ուղղի համար ներմուծվել և ուսումնասիրվել է X բազմության l-իջեցման $(X \downarrow_l \hat{X})$ գաղափարը: Այն ստանում ենք X-ից հեռացնելով l-ի հետ X-ում չհատվող մաքսիմալները և l-ի հետ մեկ հանգույցում հատվող մաքսիմալների զույգերը: Ի տարբերություն նախկինում կիրառվող մեթոդի այս դեպքում առաջին քայլում X-ից հեռացվում են միայն սկզբնական բազմության մաքսիմալ ուղիղները: Սահմանվել է նաև կանոնավոր ուղղի գաղափարը: Դա այն l ուղիղն է, որը $\hat{X}$ բազմության համար հանդիսանում է մաքսիմալ: Նշված գաղափարները հնարավորություն են տալիս շատ ավելի պարզորոշ բնութագրել X բազմության այն հանգույցների ենթաբազմությունը, որոնք օգտագործում են l ուղիղը, ինչպես նաև գտնել այդ ենթաբազմության հզորությունը:

Ապացուցվել են նաև GC_n բազմությունների նոր հատկություններ, որոնք վերաբերում են n-հանգույցանի ուղիղներին: Բացի այդ կատարվել են GC_n բազմությունների հայտնի հիմնական հատկությունների մի շարք ճշգրտումներ:

Ըստ Գասքա-Մաեզթուի վարկածի (1982թ.)՝ յուրաքանչյուր GC_n բազմության համար գոյություն ունի առնվազն մեկ մաքսիմալ ուղիղ: Մինչ այժմ այդ վարկածը ապացուցվել է միայն $n \leq 5$ դեպքերի համար:

Աշխատանքի վերջին՝ հինգերորդ գլխում կատարվել է կարևոր քայլ Գասքա-Մաեզթուի վարկածը $n = 6$ դեպքում ապացուցելու համար: Այն է՝ ապացուցվել է որ բազմության ոչ մի հանգույցի ֆունդամենտալ բազմանդամ չունի *(6,6,6,4,3,2)* մաքսիմալ բաշխման հաջորդականություն: Նշենք, որ այս փաստի անալոգը Գասքա-Մաեզթուի վարկածի $n = 5$-ի դեպքի ապացույցում ունեցել է վճռորոշ դեր:

Այստեղ ապացուցվել են նաև 3-ից մինչև 6 հանգույցանի ուղիղների օգտագործումների վերաբերյալ պնդումներ, որոնք հիմք են հանդիսացել *(6,6,6,4,3,2)* մաքսիմալ բաշխումը հերքելու համար և նաև կարող են օգտագործվել մնացած հնարավոր բաշխումները հերքելու համար: Նշենք նաև, որ *(6,6,6,4,3,2)* բաշխումը հերքելու շնորհիվ *6*-հանգույցանի ուղիղների օգտագործումների մաքսիմալ թիվը *10*-ից իջել է *7*-ի:

ЗАКЛЮЧЕНИЕ

В диссертации – на основе дифференциальных операторов с частными производными – исследуется пространство кратности точки пересечения двух алгебраических кривых.

Рассматриваются алгебраические кривые $p = 0$ и $q = 0$, для которых точка пересечения (x_0, y_0) является нулем порядка m_0 и n_0, соответственно. Предполагается, что кривые не имеют общей касательной.

Во второй главе, для каждого неотрицательного числа k, выводится точное число максимальных линейно независимых дифференциальных условии степени k для точки пересечения. В частности, доказано, что наибольшая степень дифференциальных условий равна $m_0 + n_0 - 2$, и в этом случае существует ровно одно линейно независимое условие. При этом отсюда получается максимальное число линейно независимых полиномиально-экспоненциальных решений степени k однородной системы дифференциальных уравнений с частными производными: $p(D)f = 0, q(D)f = 0$.

В третьей главе доказана теорема Нётера, где кратности описываются дифференциальными операторами с частными производными. Следует отметить, что – в отличие от известных версий теоремы Нётера – приведенные здесь условия являются необходимыми и достаточными. Далее доказывается обобщение теоремы Келли-Бахараха, где кратности описываются дифференциальными операторами с частными производными. Насколько нам известно, это первое обобщение теоремы Келли-Бахараха для случая кратных точек пересечения.

В четвертой главе изучаются простейшие n-корректные множества на плоскости – GC_n множества. Мы называем n-корректное множество узлов GC_n множеством, если фундаментальный многочлен каждого узла является произведением линейных множителей. Скажем, что узел использует данную прямую l, если она является множителем фундаментального многочлена этого узла. Назовем прямую k-узловой, если она проходит через точно k узлов множества X. В n-корректном множестве узлов не более чем $n + 1$ узлов являются коллинеарными, а $(n + 1)$-узловые прямые называются максимальными.

Для данной прямой l было введено и изучено понятие l-снижения $(X \downarrow_l \hat{X})$ для X. Мы получаем это множество, удаляя из X максимальные прямые, которые не пересекаются с l, и пары максимальных прямых, которые пересекаются в одном узле прямой l. В отличие от ранее пременяемого метода, в этом случае на первом шаге удаляются только максимальные прямые исходного множества X. Также было введено понятие регулярной прямой. Это та прямая l, которая является максимальной для множества $\hat{X}$. Указанные понятия позволяют более четко характеризовать подмножество узлов X, которые используют прямую l, а также найти мощность этого подмножества.

Доказаны также новые свойства множеств GC_n относительно n-узловых прямых. Кроме того, проведен ряд уточнений основных свойств множеств GC_n.

Согласно гипотезе Гаска-Маезту (1982), для каждого множества GC_n существует по крайней мере одна максимальная прямая. Сейчас эта гипотеза доказана только для случаев $n \leq 5$.

В последней, пятой главе работы сделан важный шаг для доказательства гипотезы Гаска-Маэзту в случае $n = 6$. Было доказано, что ни у одного узла множества фундаментальный многочлен не имеет максимального распределения $(6, 6, 6, 4, 3, 2)$. Следует отметить, что аналог этого факта сыграл решающую роль в доказательстве гипотезы Гаска-Маэзту для случая $n = 5$. Здесь доказаны также утверждения об использовании прямых с 3-мя до 6-тью узлами, которые являлись основанием для опровержения максимального распределения $(6, 6, 6, 4, 3, 2)$. Эти утверждения могут быть использованы также для опровержения остальных возможных распределений. Также следует отметить, что благодаря опровержению распределения $(6, 6, 6, 4, 3, 2)$, максимальное число использований прямых с 6-тью узлами уменьшилось с 10 до 7.