ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Դավիթ Միերի Մարտիրոսյան

$\mathbb{R}^n$-ում սահմանափակ ուռուցիկ մարմինների հետազոտումը հավանականային մեթոդներով

Ա.01.05 – «Հավանականությունների տեսություն
և մաթեմատիկական վիճակագրություն» մասնագիտությամբ
ֆիզիկամաթեմատիկական գիտությունների թեկնածուի
գիտական աստիճանի հայցման ատենախոսության

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Investigation of bounded convex bodies in $\mathbb{R}^n$ by probabilistic methods

SYNOPSIS

of the thesis for the degree of candidate of
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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում

Գիտական ղեկավար՝ Ֆիզ. մաթ. գիտ. դոկտոր Վ.Կ. Օհանյան

Պաշտոնական ընդդիմախոսներ՝ Ֆիզ. մաթ. գիտ. դոկտոր Յու.Ա. Կուտոյանց
Ֆիզ. մաթ. գիտ. թեկնածու /
Տեխ. գիտ. դոկտոր Հ.Ս. Սուքիասյան

Առաջատար կազմակերպություն՝ ՀՀ ԳԱԱ
Ինֆորմատիկայի և ավտոմատացման
պրոբլեմների ինստիտուտ

Պաշտպանությունը կկայանա 2024թ. մայիսի 16-ին, ժ. 15:00-ին Երևանի պետական համալսարանում գործող 050 «Մաթեմատիկա» մասնագիտական խորհրդի նիստում հետևյալ հասցեով՝ 0025, Երևան, Ալեք Մանուկյան 1:

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Մասնագիտական խորհրդի գիտական քարտուղար՝
Ֆիզ. մաթ. գիտ. դոկտոր Կ.Լ. Ավետիսյան

The topic of the thesis was approved at Yerevan State University

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Problems of NAS RA

The defence will take place on May 16, 2024, at 15:00 during the session of the Specialized Council 050 ”Mathematics” operating at Yerevan State University (1 Alek Manukyan St, Yerevan 0025).

The thesis is available at the library of Yerevan State University. The synopsis was sent on April 4, 2024.

Scientific secretary of the specialized Council
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Overview

Topicality. In the 1930s, at the University of Hamburg, W. Blaschke set the stage for exploring the application of probabilistic concepts in deriving geometrically significant results, particularly for convex bodies (see the preface of [1]). The existence of a bijection between bounded convex bodies (up to translations and reflections) and their chord length distribution functions turned out to be not always possible, even for polygons (see [2]). Subsequently, in 1986, G. Matheron [3] put forth a conjecture suggesting that, rather than relying on the chord length distribution function, the covariogram of a convex body in $\mathbb{R}^n$, or equivalently, its orientation-dependent chord length distribution (ODCLD) function, could serve as such a deterministic characteristic. Regarding this hypothesis, pivotal results emerged until 2009 (see [4]-[7]), demonstrating rejection for $n \geq 4$ but validation for all planar convex domains and three-dimensional convex polytopes. Since then, the examination of orientation-dependent chord length distribution and covariogram for diverse geometric shapes has acquired significant importance within the scope of researching the recognition of a convex body based on the distribution of characteristics within its lower-dimensional sections (for recent results see [8]-[13]). It constitutes a central task in geometric tomography, a field of mathematics introduced by R. Gardner in 2006 [14].

This thesis not only extends the mentioned line of inquiry to a novel class of convex bodies but also, in line with Blaschke's legacy, explores other stochastic models that establish connections between the geometric and probabilistic characteristics of convex bodies. The diverse nature of the models leads to various problems requiring a unique approach to overcome difficulties of different types and levels. An example that served as motivation for this work was a classical result by R. Sulanke [15] concerning the probability mass function of the number of intersection points generated by three random lines within a planar convex body. Increasing the number of lines significantly elevates the complexity of determining the aforementioned probability mass function.

Another well-known random variable involved in this research is the Euclidean distance between two random points chosen independently and uniformly from $\mathbb{D} \subset \mathbb{R}^d$ (see [1], chapter 4). Extensive research has been conducted on this random variable for various bounded bodies $\mathbb{D}$, including computation of the average distance within a cube [16] (known as Robbins constant), on the surface of a cube [17], within a hyperball [18], as well as bounding the average distance within a hypercube [19] or furthermore, within compact subsets of $\mathbb{R}^d$ with unit diameter [18]. The results known for a cube were extended to the 4th and 5th dimensions [20] but for higher dimensions the increase of algebraic complexity associated with derivation procedures was a strong limiting factor. In dimensions $d \leq 3$, closed-form expressions are obtained for the probability density function (PDF) of $D_d(\mathbb{D})$ in [21]-[26] for numerous geometric shapes of $\mathbb{D}$. For d-dimensional convex bodies, a relation between integrals for the powers of $D_d(\mathbb{D})$ and random chord length in $\mathbb{D}$ is well known (see [1], pp.

46-47). A connection between the chord length distribution of an infinitely long cylinder and that of its base is considered in [27]. Relations between the chord length distribution function of $\mathbb{D}$ and the distribution function of $D_d(\mathbb{D})$ are explored in [23] and [28], conditional moments of $D_d(\mathbb{D})$ are introduced and studied in [29]. A unified approach for determining the PDF of $D_d(\mathbb{D})$ for typical compact sets is suggested in [30]. It also provides a good list of references for related results of theoretical and applied character.

Objectives.

- Enlarge the class of convex bodies that admit an explicit representation of their OD-CLD function and the covariogram in terms of their geometric characteristics.
- For a planar convex body D, explore the relationship between the probabilities of a certain number of random lines producing a given number of intersection points inside D and a family of geometric invariants of D.
- Extend the concept of a covariogram from bounded convex bodies to the entire space $\mathbb{R}^d$ ensuring that the correspondence between the covariogram and interpoint distance, observed in bounded convex bodies, is maintained.

Research methods. Methods from Integral and Stochastic Geometry and Probability Theory are applied.

Scientific novelty. All main results are new. A summary is provided in this booklet right after the main content.

Theoretical and practical value. The main results of the work are theoretical. Possible practical applications may refer to medicine, stereology, crystallography, spatial analysis, and machine learning.

Approbation. The main results of the work were presented

- regularly, during the scientific seminars at the Chair of Probability Theory and Mathematical Statistics at Yerevan State University;
- in the AMU annual session dedicated to the 100th anniversary of the Armenian Mathematical Union, Nov 6, 2021;
- in the International Conference, Mathematics in Armenia: Advances and Perspectives, III, July 2-8, 2023;
- in the YSU university-wide sub-conference of the Center for Mathematical and Applied Research, April 2-4, 2024.

Publications. The results presented in the thesis are published in five papers, comprising four scientific articles and an abstract from an international conference thesis. The list is accessible within this booklet, positioned at the conclusion of the references section.

The structure and the content of the thesis. The thesis consists of an introduction, three chapters, a summary, and references. The number of references is 56. The thesis comprises 111 pages.

The Main Content of the Thesis

The introduction represents the literature review and the overview of the main results.

The first chapter focuses on finding an explicit representation of the ODCLD function for any right prism based on an arbitrary convex quadrilateral. Sections 1.1-1.3 showcase preliminaries and our early discoveries [36] regarding the ODCLD functions of a rectangular parallelepiped and a right prism based on a right trapezoid.

The necessary terminology to build the ODCLD function of a quadrilateral is provided in Section 1.4. In a Cartesian plane, for any convex quadrilateral **D** there are points $B(b,0),\ b > 0$, $A \in \{(x,y) : x \geq 0, y > 0\}$, and $C \in \{(x,y) : x > 0, y > 0\}$ such that **D** is congruent to the quadrilateral $OACB$, where O is the origin of coordinates. We will call such a quadrilateral **an image** of **D**. The side OB will be called **the base**, the sides OA and BC will be called **legs**, α and β will stand for the inclination angles (measured anticlockwise from the positive direction of x-axis) of the legs OA and BC, respectively. If $\alpha \leq \beta$ then the quadrilateral $OABC$ will be called **a standard image** of **D**. If α_0 and β_0 are the inclination angles of the diagonals OC and BA, respectively, then we use the notation $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ for that standard image of **D**. The values $\alpha_0, \alpha, \beta, \beta_0$ determine another parameter γ, the inclination angle of AC. We classify the standard images into two categories based on the value of γ. Due to convexity of **D**, either $0 \leq \gamma < \alpha_0$, or $\beta_0 < \gamma < \pi$. If the first inequality occurs, we will call the standard image to be of **Type 1**, otherwise - of **Type 2**.

In the upcoming text, L_n will stand for the n-dimensional Lebesgue measure, and $\mathbb{S}^n$ for the unit sphere in $\mathbb{R}^{n+1}$. Let l_φ be the subspace of $\mathbb{R}^2$ spanned by the vector $\phi = (\cos\varphi, \sin\varphi) \in \mathbb{S}^1$. By $\phi^\perp$ we denote the orthogonal complement of l_φ in $\mathbb{R}^2$. For any $y \in \phi^\perp$, let $l_\varphi + y$ be the line parallel to ϕ and passing through y. For a bounded convex set $E \subset \mathbb{R}^2$, we denote $\chi(l_\varphi + y) = L_1\big((l_\varphi + y) \cap E\big)$, and, if the line $l_\varphi + y$ has a common segment with E, then we will say that it makes a chord in E of length $\chi(l_\varphi + y)$.

Let $\Pi_E(\varphi)$ be the orthogonal projection of E onto $\phi^\perp$. Assuming that y is uniformly distributed over $\Pi_E(\varphi)$, the ODCLD function in direction ϕ for E is defined by

$$F_E(x,\varphi) = \frac{L_1(\Pi_E^x(\varphi))}{b_E(\varphi)},$$

where $\Pi_E^x(\varphi) = \{y \in \Pi_E(\varphi) : \chi(l_\varphi + y) \leq x\}$ and $b_E(\varphi) = L_1(\Pi_E(\varphi))$.

Since $l_{\varphi-\pi} = l_\varphi$, we can assume $\varphi \in [0, \pi)$. To determine the ODCLD function $F_{\mathbf{D}_s}(x, \varphi)$ we use the quantities

$$x_0(\varphi) = \min_{y \in \phi_v^\perp} \chi(l_\varphi + y) \text{ and } x_1(\varphi) = \max_{y \in \phi_v^\perp} \chi(l_\varphi + y),$$

where $\phi_v^\perp$ is the set of vectors $y \in \phi^\perp$ so that the line $l_\varphi + y$ passes through a vertex of $\mathbf{D}_s$ and makes a chord of positive Lebesgue measure there. The quantity $x_1(\varphi)$ coincides with the length of the longest chord

$$x_{\max}(\varphi) = \max_{y \in \Pi_{\mathbf{D}_s}(\varphi)} \chi(l_\varphi + y),$$

and any chord of length $x_{\max}(\varphi)$ is known as a **φ-diameter** of $\mathbf{D}_s$ (see [31], p. 248). We extend this concept: in the upcoming text, where convenient, we will call it a **first-order φ-diameter** of $\mathbf{D}_s$, and any chord of length $x_0(\varphi)$ will be called a **second-order φ-diameter** of $\mathbf{D}_s$.

In addition to $x_0(\varphi)$ and $x_1(\varphi)$, we introduce three more orientation-dependent characteristics $\ell_0(\varphi)$, $\ell(\varphi)$, and $\ell_1(\varphi)$ of the standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$. These characteristics are non-negative continuous functions and satisfy to $b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi)$ for all $\varphi \in [0, \pi)$. We call them **supplementary φ-measures** of $\mathbf{D}_s$. The characteristics are defined case by case in Section 1.4. Readers, already familiar with the concept of X-ray (refer to Chapter 1 of [14]), may benefit while contemplating the origins and significance of the newly introduced orientation-dependent characteristics. To determine the ODCLD function, acquiring orientation-dependent X-rays is sufficient (see, for example, [32]). These X-rays, which exhibit convex functions with up to three graph pieces for a convex quadrilateral, can be accurately determined using φ-diameters and supplementary φ-measures as necessary parameters.

The first compressed results are presented in Section 1.5, where the ODCLD function and the covariogram of a convex quadrilateral are found in terms of the lengths of orientation-dependent diameters and supplementary measures.

Theorem 1.5.1. *Let $\boldsymbol{D}_s$ be a standard image of a convex quadrilateral $\boldsymbol{D}$ and $0 \leq \varphi < \pi$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters, and ℓ_0, ℓ, ℓ_1 are the supplementary φ-measures of $\boldsymbol{D}_s$, then*

$$F_{\mathbf{D}_s}(x, \varphi) = \frac{1}{\ell_0 + \ell + \ell_1} \begin{cases} 0, & \text{if } x < 0 \\ \left(\dfrac{\ell_0}{x_0} + \dfrac{\ell_1}{x_1}\right) x, & \text{if } 0 \leq x < x_0(\varphi) \\ \ell_0 + \dfrac{x - x_0}{x_1 - x_0}\ell + \dfrac{x}{x_1}\ell_1, & \text{if } x_0(\varphi) \leq x < x_1(\varphi) \\ \ell_0 + \ell + \ell_1, & \text{if } x \geq x_1(\varphi) \end{cases}.$$

Corollary 1.5.1. *The function $F_{\boldsymbol{D}_s}(\cdot,\varphi)$ is continuous on the real axis if and only if the φ-diameter of $\boldsymbol{D}_s$ is unique. If for some φ, the φ-diameter of $\boldsymbol{D}_s$ is not unique then $F_{\boldsymbol{D}_S}(\cdot,\varphi)$ holds a jump discontinuity at $x_{\max}(\varphi)$. The jump is equal to*

$$\frac{\ell}{\ell_0+\ell+\ell_1}.$$

Below, the notation $C_{\mathbf{D}_s}(t,\varphi)$ stands for the covariogram $C_{\mathbf{D}_s}(t\phi)$, where $t\geq 0$.

Theorem 1.5.2. *Let $\boldsymbol{D}_s$ be a standard image of a convex quadrilateral $\boldsymbol{D}$ and $0\leq\varphi<\pi$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters, and $\ell_0,\ \ell,\ \ell_1$ are the supplementary φ-measures of $\boldsymbol{D}_s$, then $C_{\mathbf{D}_s}(t,\varphi)=$*

$$=\begin{cases}\dfrac{x_0\ell_0+(x_0+x_1)\ell+x_1\ell_1}{2}-(\ell_0+\ell+\ell_1)t+\dfrac{1}{2}\left(\dfrac{\ell_0}{x_0}+\dfrac{\ell_1}{x_1}\right)t^2, & \text{if }\ 0\leq t<x_0\\ \dfrac{1}{2}\left(\dfrac{\ell_1}{x_1}+\dfrac{\ell}{x_1-x_0}\right)(x_1-t)^2, & \text{if }\ x_0\leq t<x_1\\ 0, & \text{if }\ t\geq x_1\end{cases}.$$

For a standard image $\mathbf{D}_s=[b,\alpha_0,\alpha,\beta,\beta_0]$, consider $\Lambda=\{\alpha,\ \beta\},\ \Delta=\{\alpha_0,\ \beta_0\},\ \Sigma=\{0,\ \alpha,\ \gamma,\ \beta\}$, which are the sets of the inclination angles of the legs, diagonals, and the sides of $\mathbf{D}_s$, respectively. For any $\varphi\in[0,\ \pi)$, we define the functions $X_\varphi:\Lambda\times\Delta\times\Sigma\backslash\{\varphi\}\longrightarrow\mathbb{R}$ and $L_\varphi:(\Lambda\times\Delta)\cup(\Delta\times\Lambda)\longrightarrow\mathbb{R}$ by

$$X_\varphi(x,\ y,\ z)=\frac{b\sin x\sin(y-z)}{\sin(y-x)\sin(z-\varphi)},\ \ L_\varphi(x,\ y)=\frac{b\sin(x-\varphi)\sin y}{\sin(x-y)}.$$

In Section 1.6, all five orientation-dependent characteristics, $x_0(\varphi),x_1(\varphi),\ell_0(\varphi),\ell(\varphi),\ell_1(\varphi)$ are comfortably expressed by X_φ and L_φ for both Type 1 and Type 2 images.

The last section of the first chapter is devoted to the question of finding the ODCLD function and the covariogram of a right quadrilateral prism. Denote by $\mathbf{D}_s^h$ the right prism $\{(x,y,z):(x,y)\in\mathbf{D}_s,\ 0<z\leq h\}$, where $\mathbf{D}_s$ is a standard image of a convex quadrilateral. For a vector $\omega=(\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta)\in\mathbb{S}^2$, let $\omega^\perp$ be the orthogonal complement of $\{t\omega\ :\ t\in\mathbb{R}\}$ in $\mathbb{R}^3$, and $\Pi_{\mathbf{D}_s^h}(\varphi,\theta)$ be the orthogonal projection of $\mathbf{D}_s^h$ onto the plane $\omega^\perp$.

We define the chord length distribution function in direction ω for $\mathbf{D}_s^h$ by

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta)=\frac{L_2\{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta):\chi(l_{(\varphi,\theta)}+y)\leq t\}}{b_{\mathbf{D}_s^h}(\varphi,\theta)},$$

where $l_{(\varphi,\theta)}+y$ is the line that passes through $y\in\omega^\perp$ and has direction vector ω, $\chi(l_{(\varphi,\theta)}+y)=L_1\big((l_{(\varphi,\theta)}+y)\cap\mathbf{D}_s^h\big)$, and $b_{\mathbf{D}_s^h}(\varphi,\theta)=L_2(\Pi_{\mathbf{D}_s^h}(\varphi,\theta))$.

Denote $x_{\max}(\varphi,\theta) = \max\limits_{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta)} \chi(l_{(\varphi,\theta)}+y)$.

Theorem 1.7.1. *For a $\varphi\in[0,\,\pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\boldsymbol{D}_s$, respectively. Let $\ell_0,\ \ell,\ \ell_1$ be the supplementary φ-measures of $\boldsymbol{D}_s$, and denote $b_{\boldsymbol{D}_s} = \ell_0+\ell+\ell_1$. Then, for the direction $\omega = (\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta)$, $0\le\theta\le\frac{\pi}{2}$ and the prism $\boldsymbol{D}_s^h$, the following statements take place:*

(a) *If $\tan^{-1}\frac{h}{x_0} < \theta \le \frac{\pi}{2}$ and $0\le t < x_{\max}(\varphi,\theta)$, or $0\le\theta\le\tan^{-1}\frac{h}{x_0}$ and $0\le t < x_0\sec\theta$, then*

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta) = \frac{a_1 t + a_2 t^2}{\|\mathbf{D}_s\|\sin\theta + b_{\mathbf{D}_s} h\cos\theta},$$

where

$$a_1 = h\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)\cos^2\theta + b_{\mathbf{D}_s}\sin 2\theta,\quad a_2 = -\frac{3}{2}\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)\sin\theta\cos^2\theta;$$

(b) *If $0\le\theta\le\tan^{-1}\frac{h}{x_0}$ and $x_0\sec\theta\le t < x_{\max}(\varphi,\theta)$, then $x_0 < x_1$ and*

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta) = \frac{c_0 + c_1 t + c_2 t^2}{\|\mathbf{D}_s\|\sin\theta + b_{\mathbf{D}_s} h\cos\theta},$$

where

$$c_0 = (h\cos\theta + \frac{x_0}{2}\sin\theta)\left(\ell_0 - \frac{\ell x_0}{x_1 - x_0}\right),$$

$$c_1 = (h\cos^2\theta + x_1\sin 2\theta)\left(\frac{\ell}{x_1-x_0}+\frac{\ell_1}{x_1}\right),\quad c_2 = -\frac{3}{2}\sin\theta\cos^2\theta\left(\frac{\ell}{x_1-x_0}+\frac{\ell_1}{x_1}\right).$$

Corollary 1.7.1. *Let*

$$\mu(\varphi,\theta) = L_2\left(\{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta) : \chi(l_{(\varphi,\theta)}+y) = x_{max}(\varphi,\theta)\}\right).$$

The function $F_{\boldsymbol{D}_s^h}(\cdot,\varphi,\theta)$ is continuous on the real axis if and only if $\mu(\varphi,\theta)=0$. Otherwise, if $\mu(\varphi,\theta)>0$ for some pair (φ,θ), then $F_{\boldsymbol{D}_s^h}(\cdot,\varphi,\theta)$ has a jump discontinuity at $x_{\max}(\varphi,\theta)$. The jump is equal to

$$\frac{\mu(\varphi,\theta)}{\|\mathbf{D}_s\|\sin\theta + b_{\mathbf{D}_s} h\cos\theta}.$$

Theorem 1.7.2. *For a $\varphi\in[0,\,\pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\boldsymbol{D}_s$, respectively. Let $\ell_0,\ \ell,\ \ell_1$ be the supplementary φ-measures of $\boldsymbol{D}_s$, and denote $b_{\boldsymbol{D}_s} = \ell_0+\ell+\ell_1$. Then, for the direction $\omega = (\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta)$, $0\le$*

$\theta \le \frac{\pi}{2}$, the covariogram $C_{\mathbf{D}^h_s}(t\omega) = C_{\mathbf{D}^h_s}(t, \varphi, \theta)$ of the prism $\boldsymbol{D}^h_s$ has the following representation:

***(a)** If $\tan^{-1}\frac{h}{x_0} < \theta \le \frac{\pi}{2}$ and $0 \le t < x_{\max}(\varphi, \theta)$, or $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $0 \le t < x_0 \sec\theta$, then*

$$C_{\mathbf{D}^h_s}(t, \varphi, \theta) = \left(\|\mathbf{D}_s\| - b_{\mathbf{D}_s}\cos\theta \cdot t + \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)\cos^2\theta \cdot t^2\right)(h - \sin\theta \cdot t);$$

***(b)** If $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $x_0 \sec\theta \le t < x_{\max}(\varphi, \theta)$, then $x_0 < x_1$ and*

$$C_{\mathbf{D}^h_s}(t, \varphi, \theta) = \frac{1}{2}\left(\frac{\ell}{x_1 - x_0} + \frac{\ell_1}{x_1}\right)(x_1 - \cos\theta \cdot t)^2(h - \sin\theta \cdot t).$$

Consider an open convex planar domain D with perimeter L and area F. We will assume that D contains the origin of the Cartesian plane, and for a line $g \subset \mathbb{R}^2$, we denote by (p, φ) the polar coordinates of the foot of the perpendicular from the origin onto g. Let $[D]$ be the set of all lines in $\mathbb{R}^2$ that meet D. Let $\chi(g) = g \cap D$ be the chord in D produced by the line g, and $|\chi(g)|$ be the length of $\chi(g)$.

We consider N_n, the number of intersection points of n random lines in D, given that all n lines meet D, and denote $p_{nk} = \mathbb{P}(N_n = k)$. It is easy to obtain $p_{21} = \frac{2\pi F}{L^2}$ (see, for example, [1], p. 53). The formulas for intersection probabilities p_{3k}, suggested in [1], p. 65, contain a mistake. The correct formulas are

$$p_{33} = \frac{8I_2 - U}{L^3},\ p_{32} = \frac{3U - 12I_2}{L^3},\ p_{31} = \frac{6\pi FL - 3U}{L^3},$$

established earlier by R. Sulanke in [15], where

$$I_2 = \int_{[D]} |\chi(g)|^2 dg \text{ and } U = \int_{g_1 \cap g_2 \in D} u(g_1, g_2) dg_1 dg_2,$$

where $u(g_1, g_2)$ denotes the perimeter of the convex quadrilateral verticed at the points of intersections of the lines g_1 and g_2 with the boundary ∂D. The measure element dg is interpreted as $dg = dp d\varphi$.

The second chapter of the thesis is dedicated to deriving explicit formulas for probabilities p_{4k}, where $k = 1, 2, ..., 6$, expressed in terms of newly defined invariants of D. To achieve this, we employed Ambartzumian's combinatorial algorithm ([33], chapter 5 and [34]). After preliminaries (Section 2.1), we adapted the combinatorial algorithm to the specific context in Section 2.2. Towards the end of the section, Sulanke's formulas are reproduced effortlessly.

New invariants are introduced in Section 2.3, then p_{46} and p_{45} are computed.

Definition 2.3.1. *For any $g_1 \cap g_2 \in D$ we define*

$$d(g_1, g_2) = |\chi(g_1)| + |\chi(g_2)|, \quad c(g_1, g_2) = \mu\big([\chi(g_1)] \cap [\chi(g_2)]\big),$$

$$u(g_1, g_2) = \big|\partial\big(conv\big(\cup_{i=1}^{2} g_i \cap D\big)\big)\big|,$$

and for any three lines g_1, g_2, g_3 such that $g_i \cap g_j \in D$, $1 \le i < j \le 3$ we define

$$v(g_1, g_2, g_3) = \big|\partial\big(conv\big(\cup_{i=1}^{3} g_i \cap D\big)\big)\big|,$$

where $conv(X)$ denotes the convex hull of $X \subset \mathbb{R}^2$, and $|\partial Y|$ denotes the perimeter of a convex domain Y.

The new definition of $u(g_1, g_2)$ coincides with the one we have used so far.

Along with the well-known invariants $I_k = \int_{[D]} |\chi(g)|^k dg, \ k = 0, 1, 2, \ldots$ let's consider new ones:

$$D_k = \int_{g_1 \cap g_2 \in D} d^k(g_1, g_2) dg_1 dg_2, \quad C_k = \int_{g_1 \cap g_2 \in D} c^k(g_1, g_2) dg_1 dg_2,$$
$$U_k = \int_{g_1 \cap g_2 \in D} u^k(g_1, g_2) dg_1 dg_2, \quad V_k = \int_{g_i \cap g_j \in D,\, 1 \le i < j \le 3} v^k(g_1, g_2, g_3) dg_1 dg_2 dg_3.$$

Theorem 2.3.1.

$$p_{46} = \frac{3U_2 + 9C_2 - 12D_2 + 4V_1}{4L^4}, \quad p_{45} = \frac{36D_2 - 9U_2 - 15C_2 - 12V_1}{2L^4}.$$

The remaining probabilities p_{4k}, $k \le 4$ are computed in Section 2.4. Given $g_1 \cap g_2 \in D$, we denote by $\rho_1, \rho_2, \rho_3, \rho_4$ the lengths of four consecutive sides of the quadrilateral $conv\big((g_1 \cup g_2) \cap \partial D\big)$. To avoid ambiguity, we will always assume that the first two sides lie in different half-planes with respect to g_1. If two lines, e.g. g_2 and g_3, are from $[D]$ but do not meet inside D, then d_1, d_2 will stand for the lengths of the diagonals of $conv\big((g_2 \cup g_3) \cap \partial D\big)$, and s_1, s_2 will represent the lengths of the sides of the quadrilateral which are different from $\chi(g_2), \chi(g_3)$.

We extend the set of invariants of D by three more:

$$R = \int_{g_1 \cap g_2 \in D} \big((\rho_1 + \rho_2)(\rho_3 + \rho_4) + (\rho_2 + \rho_3)(\rho_4 + \rho_1)\big) dg_1 dg_2,$$

$$Q_s = \int_{g_2 \cap g_3 \notin D} (s_1 + s_2)(d_1 + d_2 - s_1 - s_2) dg_2 dg_3,$$

$$Q_d = \int_{g_2 \cap g_3 \notin D} (d_1 + d_2)(d_1 + d_2 - s_1 - s_2) dg_2 dg_3.$$

Theorem 2.4.1. *Let $p_{44}^{(1)}$ be the probability that $g_1, g_2, g_3, g_4 \in [D]$ produce 4 intersection points inside D and some three of them intersect each other inside D. Then $p_{44} = p_{44}^{(1)} + p_{44}^{(2)}$, where*

$$p_{44}^{(1)} = \frac{6}{L^4}(2V_1 - 4D_2 + C_2 + U_2) \text{ and } p_{44}^{(2)} = \frac{3}{L^4}\left(\frac{3U_2 + C_2}{2} - 8I_3 - 2R - Q_s\right).$$

Three intersection points made by four lines from $[D]$ can occur in three ways:
Event 1: The lines produce three chords each possessing two intersection points, and one containing no intersection point;
Event 2: The lines produce two chords each possessing 2 intersection points, and the other two each possessing 1 intersection point;
Event 3: The lines produce three chords each possessing 1 intersection point, and one possessing 3 intersection points.

We denote by $p_{43}^{(1)}$, $p_{43}^{(2)}$, $p_{43}^{(3)}$ the probabilities of Event 1, Event 2, and Event 3, respectively.

Theorem 2.4.2.

$$p_{43} = p_{43}^{(1)} + p_{43}^{(2)} + p_{43}^{(3)},$$

where

$$p_{43}^{(1)} = \frac{4}{L^4}(C_1 L - V_1), \;\; p_{43}^{(2)} = \frac{12}{L^4}(Q_s + 2R - U_2),$$

$$p_{43}^{(3)} = \frac{3}{L^4}(C_2 - 4D_2 - U_2) + \frac{4}{L^4}(Q_d + 2R) + \frac{64}{L^4} I_3.$$

Two intersection points generated by four lines are possible in two scenarios:
Event 1: One chord possesses 2 intersection points, two of the chords possess 1 intersection point each, and one chord does not possess any intersection point;
Event 2: Each chord of the four lines possesses exactly 1 intersection point.
Let the probabilities of the above mentioned events be $p_{42}^{(1)}$ and $p_{42}^{(2)}$, respectively.

Theorem 2.4.3.

$$p_{42} = p_{42}^{(1)} + p_{42}^{(2)},$$

where

$$p_{42}^{(1)} = 4p_{32} - \frac{3}{L^4}\big(C_2 - 4D_2 - U_2 - 4(Q_s + 2R)\big),$$

$$p_{42}^{(2)} = \frac{12\pi^2 F^2}{L^4} + \frac{48 I_3}{L^4} - \frac{3}{4L^4}\big(U_2 - C_2 + 12D_2\big) - \frac{1}{4}\big(p_{44}^{(1)} + 2p_{43}^{(2)}\big).$$

Theorem 2.4.4.

$$p_{41} = 2p_{31} - 2p_{42}^{(2)} - \frac{6U_1}{L^3} + \frac{6U_2}{L^4}.$$

Finally, the probability of having no intersection points inside D is $p_{40} = 1 - \sum_{k=1}^{6} p_{4k}$.

In Section 2.5, we expressed the invariants V_1 and I_3 by intersection probabilities. The value of I_3 is known (by Crofton, [1], p. 47) to be equal to $3F^2$ but we did not need it below.

Theorem 2.5.1. *The following identities hold:*

$$V_1 = L^4\big(p_{33} - \frac{1}{4}p_{43}^{(1)}\big), \;\; I_3 = \frac{L^4}{32}\big(4p_{33} + p_{43}^{(3)} - p_{43}^{(1)}\big).$$

The formulas obtained for intersection probabilities motivated us to compute invariants of D through simulations. For example, we used Python 3 software to approximate the values of I_2, U_1, I_3, and V_1 for the unit disk. Simulations code can be found here: `http://rb.gy/1wei7h`.

Expressions of all the new invariants in terms of r for a disc of radius r are established in Section 2.6. As a result, the following theorem is proved.

Theorem 2.6.1. *If D is a disc with radius r then*

$$p_{46} = \frac{1}{4} - \frac{17}{8\pi^2}, \;\; p_{45} = \frac{29}{8\pi^2} - \frac{1}{4},$$

$$p_{44} = \frac{43}{4\pi^2} - \frac{7}{8}, \; p_{44}^{(1)} = \frac{23}{2\pi^2} - 1, \; p_{44}^{(2)} = \frac{1}{8} - \frac{3}{4\pi^2},$$

$$p_{43} = 1 - \frac{29}{4\pi^2}, \; p_{43}^{(1)} = \frac{23}{4\pi^2} - \frac{1}{2}, \; p_{43}^{(2)} = 1 - \frac{35}{4\pi^2}, \; p_{43}^{(3)} = \frac{1}{2} - \frac{17}{4\pi^2},$$

$$p_{42} = \frac{7}{4} - \frac{121}{8\pi^2}, \; p_{42}^{(1)} = \frac{3}{2} - \frac{13}{\pi^2}, \; p_{42}^{(2)} = \frac{1}{4} - \frac{17}{8\pi^2},$$

$$p_{41} = \frac{29}{8\pi^2} - \frac{1}{4}, \;\; p_{40} = \frac{13}{2\pi^2} - \frac{5}{8}.$$

For a bounded body $\mathbb{D} \subset \mathbb{R}^d$, consider the Euclidean distance between two random points chosen independently and uniformly from $\mathbb{D}$. We denote it by $D_d(\mathbb{D})$.

Along with $D_d(\mathbb{D})$, let us consider the covariogram of $\mathbb{D}$, denoted by $C_{\mathbb{D}}(\boldsymbol{t})$, where $\boldsymbol{t} \in \mathbb{R}^d$. When $\mathbb{D}$ is a bounded convex body with a non-empty interior in $\mathbb{R}^d$, then the two considered characteristics of $\mathbb{D}$ are interrelated as follows:

$$f_{D_d(\mathbb{D})}(h) = \frac{h^{d-1}}{L_d^2(\mathbb{D})} \int_{\mathbb{S}^{d-1}} C_{\mathbb{D}}(h\boldsymbol{u})d\boldsymbol{u},\ h > 0, \tag{1}$$

where $\mathbb{S}^{d-1}$ is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$, centered at the origin, and $L_d(\mathbb{D})$ is Lebesgue d-measure of $\mathbb{D}$.

In the third chapter of the thesis, we extended the concepts of covariogram $C_{\mathbb{D}}$ and interpoint distance $D_d(\mathbb{D})$ from bounded convex bodies to the entire space $\mathbb{R}^d$ and established a relation between them.

The first problem that arises in our way is the nature of randomness of choosing a point from $\mathbb{D} = \mathbb{R}^d$. The uniform distribution is no longer applicable to this case and therefore we naturally replace it with a multivariate normal distribution.

The second obstacle lies in the challenge of applying the language and sense of geometry to define the covariogram of $\mathbb{R}^d$. We will define it analytically based on the following observation. If $\mathbb{D}$ is a convex body and $\mathcal{P}_1, \mathcal{P}_2$ are chosen uniformly and independently from $\mathbb{D}$, then it is easy to check (see, for example, [26]) that

$$f_{\mathcal{P}_1 - \mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{L_d^2(\mathbb{D})},$$

which can be equivalently written as

$$f_{\mathcal{P}_1 - \mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbb{D}}(\boldsymbol{t})}{C_{\mathbb{D}}^2(\boldsymbol{0})}. \tag{2}$$

Thus, the covariogram should be a positive function defined on the entire space that satisfies (2).

If $\mathbf{X}$ is a d-variate normal random vector having mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ then we will denote this condition by $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We denote $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_d]^T$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d > 0$ are the eigenvalues of $\boldsymbol{\Sigma}$. We assume $\boldsymbol{\mu} = \boldsymbol{0}$ and the diagonal of $\boldsymbol{\Sigma}$ consisting of 1s. If $\boldsymbol{X_1}, \boldsymbol{X_2} \sim N_d(\boldsymbol{0}, \boldsymbol{\Sigma})$ are independent, we denote $D_d = \|\boldsymbol{X_1} - \boldsymbol{X_2}\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

After preliminaries, in Section 3.2, we independently addressed the scenario of uncorrelated coordinates and deduced the density function and moments of the interpoint distance, drawing upon a result obtained in [35], p. 95. We easily established that if $\boldsymbol{\Sigma} = \boldsymbol{I}_d$ then $D_d \sim GG(2, d, 2)$, where $\boldsymbol{I}_d$ is the identity $d \times d$ matrix, and $GG(a, d, p)$ is the generalized Gamma distribution. Since the moments of the generalized Gamma distribution were known, as a corollary, for moments of D_d, we immediately concluded that if $\mathbf{\Sigma} = \boldsymbol{I}_d$, then

$$\mathbb{E}\left(D_d^r\right) = 2^r \frac{\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},\ r = 0, 1, 2, \dots .$$

In general, when $\mathbf{\Sigma} \neq \boldsymbol{I}_d$, even when $d = 2$, this method becomes impractical because of the demanding computations associated with complicated recursive formulas. In Section 3.3, we established new results, including integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points, characterized by correlated coordinates through a covariance matrix.

Theorem 3.3.1. *Let $F_{D_d}(\mathbf{\Sigma}, \cdot)$ be the distribution function of D_d and $\mathcal{E}_d(\boldsymbol{\lambda}, R)$ be the ellipsoid*

$$\{\boldsymbol{y} = [y_1, y_2, \dots, y_d]^T : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_d y_d^2 \leq R^2\}.$$

Then

$$F_{D_d}(\mathbf{\Sigma}, R) = \frac{1}{(2\sqrt{\pi})^d} \int_{\mathcal{E}_d(\boldsymbol{\lambda}, R)} \exp\left(-\frac{1}{4}\boldsymbol{y}^T\boldsymbol{y}\right) d\boldsymbol{y},\ R > 0.$$

Corollary 3.3.1. *The probability density function of D_d is representable as follows:*

$$f_{D_d}(\mathbf{\Sigma}, R) = \frac{R^{d-1}}{(2\sqrt{\pi})^d |\mathbf{\Sigma}|^{1/2}} \int_{\mathbb{S}^{d-1}} \exp\left(-\frac{R^2}{4}\boldsymbol{u}^T\mathbf{\Sigma}^{-1}\boldsymbol{u}\right) d\boldsymbol{u}.$$

As an application of the obtained integral representations, in Section 3.4, we determined the probability density function of the Euclidean distance between two bivariate Gaussian points in the case when there is an inter-coordinate correlation ρ.

Theorem 3.4.1. *If* $\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, *then*

$$f_{D_2}(\mathbf{\Sigma}, R) = \frac{Re^{-\frac{R^2}{4|\mathbf{\Sigma}|}}}{2\sqrt{|\mathbf{\Sigma}|}} I_0\left(\frac{\rho R^2}{4|\mathbf{\Sigma}|}\right),$$

where

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{((2k)!!)^2}$$

is the modified Bessel function of the first kind of order zero.

As another application, we established lower and upper bounds for the moments of D_d in terms of the largest and the smallest eigenvalues of the covariance matrix.

Theorem 3.4.2. *Let $\mathbb{E}\left(D_d^r\right)$ be the r-th moment of D_d. Then*

$$\frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{\lambda_d^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}} \leq \mathbb{E}\left(D_d^r\right) \leq \frac{2^r\Gamma\left(\frac{d+r}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{\lambda_1^{\frac{d+r}{2}}}{|\mathbf{\Sigma}|^{1/2}},\ r = 0, 1, 2, \dots .$$

Finally, in Section 3.5, we defined the normal covariogram of $\mathbb{R}^d$ and established an analogous relationship to (1).

Definition 3.5.1. *Let* $\mathcal{P}_1,\ \mathcal{P}_2 \sim N_d(\mathbf{0},\ \mathbf{\Sigma})$ *be independent and* $f_{\mathcal{P}_1-\mathcal{P}_2}$ *be the probability density function of* $\mathcal{P}_1 - \mathcal{P}_2$*. The function* $C_{\mathbf{\Sigma}} : \mathbb{R}^d \to (0,\ +\infty)$ *that satisfies to*

$$f_{\mathcal{P}_1-\mathcal{P}_2}(\boldsymbol{t}) = \frac{C_{\mathbf{\Sigma}}(\boldsymbol{t})}{C^2_{\mathbf{\Sigma}}(\mathbf{0})},$$

will be called the normal covariogram of $\mathbb{R}^d$ *associated with* $\mathbf{\Sigma}$.

Taking into account (1), the following theorem argues that the normal covariogram naturally extends the concept of covariogram.

Theorem 3.5.1. *Let* $\mathbf{\Sigma}$ *be the covariance matrix of a non-singular* d*-variate normal distribution and* $C_{\mathbf{\Sigma}}$ *be the normal covariogram of* $\mathbb{R}^d$ *associated with* $\mathbf{\Sigma}$*. Then*

$$C_{\mathbf{\Sigma}}(\boldsymbol{t}) = (2\sqrt{\pi})^d |\mathbf{\Sigma}|^{1/2} \exp\left(-\frac{1}{4}\boldsymbol{t}^T \mathbf{\Sigma}^{-1}\boldsymbol{t}\right),\ \boldsymbol{t} \in \mathbb{R}^d$$

and

$$f_{D_d}(\mathbf{\Sigma}, R) = \frac{R^{d-1}}{C^2_{\mathbf{\Sigma}}(\mathbf{0})} \int_{\mathbb{S}^{d-1}} C_{\mathbf{\Sigma}}(R\boldsymbol{u}) d\boldsymbol{u},\ \ R > 0.$$

Remark 3.5.1. *It is remarkable that* $C_{\boldsymbol{I}_d}(\boldsymbol{t}) = (2\sqrt{\pi})^d \exp\big(-\frac{1}{4}\|\boldsymbol{t}\|^2_d\big)$*. This illustrates that if* $\mathbb{R}^d$ *is considered as a space of points with uncorrelated coordinates then the covariogram of the space is naturally independent on the direction of translation.*

Summary

The main results obtained in the thesis are presented in three chapters.

- *For any convex quadrilateral, the notions of first and second-order* φ*-diameters, along with three supplementary measures, are introduced and evaluated for each direction* φ*. In terms of these five characteristics, explicit representations of the orientation-dependent chord length distribution (ODCLD) function and the covariogram are established for any convex quadrilateral and any right prism based on it. Continuity criteria for the ODCLD functions are established per direction.*

- *Based on R. Ambartzumian's combinatorial algorithm, a new approach is suggested for finding the probabilities* p_{nk} *of* n *random lines producing* k *intersection points inside a given convex planar domain* D*. A family of geometric characteristics of* D*, invariant under Euclidean motions, is found such that the probabilities* p_{4k} *are expressed in terms of those invariants. When* D *is a disc with a radius of* r*, the simplest forms of the invariants are derived, and the exact numerical values of* p_{4k} *are computed.*

- *Integral representations for the distribution and probability density functions of the Euclidean distance between two independent* d*-dimensional Gaussian points with correlated coordinates governed by a covariance matrix are obtained. In addition to other applications, the concept of a covariogram is extended to the entire space* $\mathbb{R}^d$ *such that the correspondence between the covariogram and interpoint distance, observed in bounded convex bodies, is maintained.*

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Ամփոփում

Ատենախոսությունը նվիրված է $\mathbb{R}^n$-ում ուռուցիկ մարմինների երկրաչափական և հավանականային բնութագրիչների կապերի ուսումնասիրությանը՝ տվյալ մարմնին առնչվող տարբեր պատահական մեծությունների ներգրավմամբ։ Ստացվել են հետևյալ հիմնական արդյունքները.

- Կամայական ուռուցիկ քառանկյան համար ներմուծվել են առաջին և երկրորդ կարգի φ - տրամագծերի և երեք լրացուցիչ չափումների գաղափարները, որոնք հաշվվել են յուրաքանչյուր φ ուղղության համար։ Ինչպես ուռուցիկ քառանկյան, այնպես էլ այդ հիմքն ունեցող ուղիղ պրիզմաների համար, ստացվել են ուղղությունից կախված լարի երկարության բաշխման (ODCLD) ֆունկցիայի և կովարիոգրամի բացահայտ ներկայացումներ՝ արտահայտված նշված հինգ բնութագրիչներով։ Տրված ուղղության համար ստացվել են ODCLD ֆունկցիաների անընդհատության հայտանիշներ։
- Ռ. Համբարձումյանի կոմբինատոր ալգորիթմի կիրառմամբ առաջարկվել է նոր մոտեցում՝ հաշվելու D հարթ ուռուցիկ տիրույթի ներսում n պատահական ուղիղների՝ k հատման կետեր առաջացնելու հավանականությունը։ Գտնվել է D-ի՝ էվկլիդյան շարժումների նկատմամբ ինվարիանտ երկրաչափական բնութագրիչների մի ընտանիք, այնպիսին, որ p_{4k} հավանականությունները արտահայտվեն այդ ինվարիանտներով։ Այն դեպքում, երբ D-ն r շառավղով շրջան է, նշված ինվարիանտների պարզագույն արտահայտությունները և p_{4k} հավանականությունների ճշգրիտ թվային արժեքները գտնվել են։
- d-չափանի գաուսյան երկու անկախ կետերի միջև եղած էվկլիդյան հեռավորության բաշխման և հավանականային խտության ֆունկցիաների համար ստացվել են ինտեգրալային ներկայացումներ, որտեղ հաշվի է առնվել պատահական կետի կոորդինատների միջև գործող հնարավոր կորելյացիան՝ տրված կովարիացիոն մատրիցով։ Այլ կիրառությունների հետ մեկտեղ, կովարիոգրամի գաղափարն ընդլայնվել է այնպես, որ $\mathbb{D}$ սահմանափակ ուռուցիկ մարմնի կովարիոգրամի և միջկետային հեռավորության միջև գործող կապը պահպանվի նաև $\mathbb{D} = \mathbb{R}^d$ դեպքում։

Резюме

Диссертация посвящена изучению связей между геометрическими и вероятностными характеристиками выпуклых тел с привлечением различных случайных величин, касающихся данного тела. Получены следующие основные результаты:

- Для произвольного выпуклого четырехугольника введены понятия φ-диаметров первого и второго порядков, а также три дополнительные измерения, которые вычислены для любого направления φ. В терминах этих пяти характеристик, установлены явные представления функции распределения длины хорды зависящей от ориентации (ODCLD) и ковариограммы для выпуклого четырехугольника и прямой выпуклой призмы с четырехугольным основанием. Установлены критерии непрерывности для функций ODCLD при заданном направлении.

- Путем применения комбинаторного алгоритма Р. Амбарцумяна предложен новый подход к нахождению чисел p_{nk}, представляющих вероятность, что n случайных прямых образуют ровно k точек пересечения внутри заданной выпуклой плоской области D. Обнаружено семейство геометрических характеристик D, инвариантных относительно евклидовых движений, таких, что вероятности p_{4k} выражаются через эти инварианты. В случае, когда D представляет собой диск радиуса r, найдены простейшие выражения для упомянутых инвариантов и точные числовые значения вероятностей p_{4k}.

- Получены интегральные представления для функций распределения и плотности вероятности евклидового расстояния между двумя независимыми d-мерными гауссовскими точками с возможной корреляцией координат, определенной ковариационной матрицей. Помимо других приложений, концепция ковариограммы расширена таким образом, чтобы связь, действующая между ковариограммой и межточечным расстоянием ограниченного выпуклого тела $\mathbb{D}$, сохранялась при $\mathbb{D} = \mathbb{R}^d$.