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Basis properties for orthogonal spline systems.

SYNOPSIS

of the thesis for the degree of candidate of
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Ատենախոսության թեման հաստատվել է ՀՀ Գիտությունների Ազգային Ակադեմիայի Մաթեմատիկայի ինստիտուտում

Գիտական ղեկավար՝ Ֆիզ. մաթ. գիտ. դոկտոր Կ.Ա. Քեռյան

Պաշտոնական ընդդիմախոսներ՝ Ֆիզ. մաթ. գիտ. դոկտոր Մ. Փասսենբրուններ

Ֆիզ. մաթ. գիտ. դոկտոր Մ.Գ. Գրիգորյան

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The thesis is available at the library of Yerevan State University.

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Scientific secretary of specialized council
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Overview

Relevance of the topic.

The trigonometric system is the first orthogonal systems of functions. It has played an important role in various branches of mathematics (harmonic analysis, number theory, mathematical physics, etc.). It is well known that the Fourier series of a continuous function can be divergent (see e.g. [2]). In 1910 A. Haar [41] constructed an orthonormal system such that the Fourier series of any continuous function f with respect to that system uniformly converges to f. Nevertheless, the Haar system does not form a basis for $C[0,1]$, since its functions are discontinuous. The first example of an orthonormal basis for $C[0,1]$ was constructed by Ph. Franklin in 1928 ([17]). The Franklin system is a complete orthonormal system of continuous, piecewise linear functions (with dyadic knots). It is obtained by applying the Gram-Schmidt orthogonalization process to the Faber-Schauder system.

The systematic investigations of the Franklin system have been started by Z. Ciesielski with his remarkable papers [11] and [12]. Since then, the Franklin system has been studied by many authors from different perspectives. The basic properties of this system, including exponential estimates for the Franklin functions and L^p-stability on dyadic blocks, have been obtained by Z. Ciesielski in [11] and [12]. These properties turned out to be an important tool in further investigations of the Franklin system. It is known that this system is a basis in $C[0,1]$ and L^p for $1 \le p < \infty$. The unconditionality of the Franklin system in $L^p, 1 < p < \infty$, has been proved by S. V. Bochkarev in [4]. Moreover, the Franklin system is an unconditional basis in all reflexive Orlicz spaces ([3]). The existence of an unconditional basis in H^1 has been first proved by B. Maurey [42], but the proof was non-constructive. The first explicit construction of an unconditional basis in H^1 is due to L. Carleson [8]. Then, P. Wojtaszczyk has obtained a characterization of the BMO space in terms of the coefficients of a function in the Franklin system and proved that the Franklin system is an unconditional basis in the real Hardy space H^1 [56]. The unconditionality of the Franklin system in real Hardy spaces $H^p, 1/2 < p \le 1$, has been obtained by P. Sjölin and J. Strömberg ([52]).

The Franklin system has had important applications in various problems of analysis. In particular, the constructions of bases in spaces $C^1(I^2)$ (see [13], [51]) and $A(D)$ (see [3]) are based on this system. Here $C^1(I^2)$ is the space of all continuously-differentiable functions $f(x,y)$ on the square $I^2 = [0,1] \times [0,1]$ with the norm

$$\|f\| = \max |f(x,y)| + \max \left|\frac{\partial f}{\partial x}\right| + \max \left|\frac{\partial f}{\partial y}\right|,$$

and $A(D)$ denotes the space of analytic functions on the open disc $D = \{z : |z| < 1\}$ that are continuously extendable up to the boundary. The norm of a function $f \in A(D)$ is defined by

$$\|f\| = \max_{|z| \le 1} |f(z)|.$$

The questions of existence of bases in $C^1(I^2)$ and $A(D)$ were posed by S. Banach [1].

Both Franklin and Haar are special cases of the orthogonal spline systems. Spline functions are piecewise polynomial functions. These functions are mostly used in problems that include some kind of interpolation. Splines are defined by a knot sequence and a degree, which is the highest degree of the polynomial that can be used in each segment of the spline. In the literature instead of degree we encounter the term order. It is the degree plus one. For example, Franklin system, which is an orthonormal system of spline functions, has order $k = 2$. After thorough examination of Franklin and Haar systems, researchers started to gradually generalize the results to the orthonormal spline systems of arbitrary order k. A celebrated result of A. Shadrin [50] states that if a sequence of knots is dense in $[0,1]$, then the orthogonal projection operator onto the space of polynomial splines of order k is bounded on $L^\infty[0,1]$ by a constant that depends only on the spline order k. As a consequence, non-periodic orthonormal spline system is a Schauder basis in $L^p[0,1]$, $1 \leq p < \infty$ and in the space of continuous functions $C[0,1]$. Moreover, Z. Ciesielski [14] obtained several consequences of Shadrin's result, one of them being an estimate for the inverse of the B-spline Gram matrix. Using this estimate, G. G. Gevorkyan and A. Kamont [28] extended a part of their result from [27] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0,1]$. Further extension required more precise estimates for the inverse of B-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained by M. Passenbrunner and A. Yu. Shadrin [44]. Using these estimates, M. Passenbrunner [43] proved that for each dense sequence of knots, the corresponding orthonormal spline system of order k is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$.

The primary focus of this thesis is to extend the existing results concerning basis properties of periodic orthonormal spline systems in the space $H^1(\mathbb{T})$. Before discussing these extensions, we mention the main results related to this topic. The periodic analogue of Shadrin's theorem is proved in [45]. In case of dyadic knots, J. Domsta [16] obtained exponential decay for the inverse of the Gram matrix of periodic B-splines, which was used to prove the unconditionality of the periodic orthonormal spline systems with dyadic knots in L^p for $1 < p < \infty$. In [38] it was proved that for any admissible point sequence the corresponding periodic Franklin system forms an unconditional basis in $L^p[0,1]$, $1 < p < \infty$. K. Keryan and M. Passenbrunner [39] obtained an important estimate for general periodic orthonormal spline functions. By combining the estimate with the methods developed in [26], they were able to derive the unconditionality of periodic orthonormal spline systems in $L^p(\mathbb{T})$, $1 < p < \infty$. Another contribution to the study of periodic orthonormal spline systems was made by M. P. Poghosyan and K. A. Keryan in [49], where they provided a geometric characterization of knot sequences for which the corresponding general periodic Franklin systems is a basis or unconditional basis in $H^1(\mathbb{T})$.

In this thesis basis and unconditional basis properties are considered for orthonormal spline systems of arbitrary order k corresponding to "regular" knots in the space of $H^1(\mathbb{T})$. A simple case of orthonormal spline system is Franklin system. It is orthonormal spline systems of order $k = 2$ corresponding to dyadic sequence of knots.

Another direction we pursued in our research was the study of uniqueness properties of multiple Franklin series. Cantor [7] (see also [2, Ch. 1, §70]) proved that the empty set is a U-set for the trigonometric system. This theorem marked the beginning of the study of uniqueness of orthogonal series. One of the important generalizations of Cantor's theorem is Vallee-Poussin's theorem [55] (see also [2, Ch. 14, § 4]): any countable set is a VP-set for a trigonometric system. Research on the uniqueness of the trigonometric series continues to this day (see [40]).

The study of the uniqueness of series for the Haar, Walsh systems and their generalizations began with the works [34], [35], [46] and [53] and continues to this day (see [30], [31], [47], and [48]).

The study of U-sets for the Franklin systems began recently with the works [20], [21]. In [22], theorems on VP-sets of the Franklin system were proved. In particular, The empty set for the Franklin system is a VP set.

In this thesis, we focus on proving theorems regarding VP-sets of multiple Franklin series.

The aim and objectives of the thesis. The main purpose of the present thesis is to study the basis properties of orthonormal spline systems in $H^1(\mathbb{T})$. The following are the main goals:

1. To characterize the knot sequences for which the corresponding orthonormal spline system of arbitrary order k is a basis in $H^1(\mathbb{T})$.

2. To demonstrate that, under certain regularity conditions on the knot sequences corresponding to an orthonormal spline system of arbitrary order k, the system serves as an unconditional basis in the Hardy atomic space on the torus $\mathbb{T}$.

3. To establish that it is necessary for the knot sequences to satisfy certain regularity conditions for the corresponding orthonormal spline system of arbitrary order k to form an unconditional basis in the Hardy atomic space on the torus $\mathbb{T}$.

4. To study of uniqueness sets for multiple Franklin systems by considering convergence in the sense of Pringsheim.

Research methods. In the thesis the methods of real analysis, functional analysis, harmonic analysis, and mathematical analysis are used.

Scientific novelty. All of the main results are new. The results are listed below:

1. If a k-admissible sequence of knots (s_n) satisfies k-regularity on the torus $\mathbb{T}$ with some parameter $\gamma \geq 1$ then the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ of the order k is basis in $H^1(\mathbb{T})$. Conversely, If periodic orthonormal spline system $(\hat{f}_n^{(k)})$ is a basis in $H^1(\mathbb{T})$ then the corresponding sequence of knots satisfy k-regularity condition with some parameter $\gamma \geq 1$.

2. The $(k-1)$-regularity condition on a k-admissible sequence of knots (s_n) on $\mathbb{T}$ is sufficient for the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ to form an unconditional basis in $H^1(\mathbb{T})$.

3. Let (s_n) be a k-admissible sequence of points on the torus $\mathbb{T}$. If the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ forms an unconditional basis in $H^1(\mathbb{T})$, then (s_n) satisfies the $(k-1)$-regularity condition on $\mathbb{T}$ for some parameter $\gamma \geq 1$.

4. If the multiple Franklin series converge in the sense of Pringsheim to a finite integrable function, except possibly on a certain sparse set, then the series is the Fourier-Franklin series of the limiting function. Moreover, under certain technical condition and the convergence of the iterated limits of the multiple Franklin series to an everywhere finite integrable function, the series is identified as the Fourier-Franklin series of the limiting function.

Theoretical and practical value. All the results and methods represent theoretical value. The methods are applied and can be extended to be further applied in theories of orthogonal series and harmonic analysis.

It is proved that under some regularity condition on the knot sequences, the corresponding periodic orthonormal spline systems are basis or unconditional basis in $H^1(\mathbb{T})$.

Additionally, we prove that sparse sets, which are defined as Cartesian product of measure zero sets, are VP-sets for multiple Franklin system.

Publications. The main results of the thesis have been published in 4 scientific articles. The list of the articles is given at the end of the Synopsis.

The structure and the volume of the thesis. The thesis consists of introduction, 4 chapters, a conclusion and a list of references. The number of references is 60. The volume of the thesis is 76 pages.

The Main Content of the Thesis

In Introduction we recall several results concerning the basis properties of different orthogonal systems in various spaces. Results on basis properties for series by Haar, Franklin or more generally orthonormal spline systems of arbitrary order have been obtained by S.-Y.A. Chang, L. Carleson, Z. Ciesielski, G. Gevorkyan, A. Kamont, K. Keryan, M. Passenbrunner, M. Poghosyan, P. Sjölin, J.O. Strömberg, P. Wojtaszczyk and others in the papers [8], [10], [28], [29], [49], [52], and [56].

Next, we provide some background on the study of uniqueness sets with respect to various orthonormal spline systems, including trigonometric, Haar, Walsh, and Franklin systems. This line of research began with Cantor's work on the trigonometric system [7], which was later generalized by Vallee-Poussin [55]. Subsequent contributions to this field were made by researchers such as G. Gevorkyan, F. G. Harutyunyan, G. Kozma, K. A. Navasardyan, A. Olevskii, M. B. Petrovskaya, M. G. Plotnikov, Yu. A. Plotnikova, A. A. Talalyan and others. The following are relevant papers on this topic [30], [31], [34], [35], [40], [46], [47], [48] and [53].

To formulate some of these results, let us begin by providing the key definitions.

Assume that $k \geq 2$ is an integer. Let $\mathcal{T} = (t_n)_{n=2}^{\infty}$ be a dense sequence of points in $[0,1]$ such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called k-admissible. For n in the range $-k+2 \leq n \leq 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order $n+k-1$ (or degree $n+k-2$) on the interval $[0,1]$ and $(f_n^{(k)})_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$. For $n \geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n+1}$ counting multiplicities and where the knots 0 and 1 have multiplicity k, i.e., $\mathcal{T}_n$ is of the form

$$\mathcal{T}_n = (0 = \tau_{n,-k} = \cdots = \tau_{n,-1} < \tau_{n,0} \leq \\ \leq \cdots \leq \tau_{n,n-1} < \tau_{n,n} = \cdots = \tau_{n,n+k-1} = 1).$$

In that case, we define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$ and, therefore, there exists a function $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ that is orthonormal to the space $\mathcal{S}_{n-1}^{(k)}$. See that this function $f_n^{(k)}$ is unique up to sign. The system of functions $(f_n^{(k)})_{n=-k+2}^{\infty}$ is called orthonormal spline system of order k corresponding to the sequence $(t_n)_{n=0}^{\infty}$.

We define Hardy atomic space on the interval $[0,1]$. A function $a : [0,1] \to \mathbb{R}$ is called an atom, if either $a \equiv 1$ or there exists an interval Γ such that the following conditions are satisfied:

(i) $\operatorname{supp} a \subset \Gamma$,

(ii) $\|a\|_\infty \leq |\Gamma|^{-1}$,

(iii) $\int_0^1 a(x)\,\mathrm{d}x = \int_\Gamma a(x)\,\mathrm{d}x = 0$.

Then, by definition, $H^1[0,1]$ consists of all functions f that have the representation

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some atoms $(a_n)_{n=1}^{\infty}$ and real scalars $(c_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$. The space H^1 becomes a Banach space under the norm

$$\|f\|_{H^1} := \inf \sum_{n=1}^{\infty} |c_n|,$$

where inf is taken over all atomic representations $\sum c_n a_n$ of f.

Now, we introduce regularity conditions for a sequence $\mathcal{T}$. For $n \geq 2$, $\ell \leq k$ and i in the range $-\ell \leq i \leq n-1$, we define $D_{n,i}^{(\ell)}$ to be the interval $[\tau_{n,i}, \tau_{n,i+\ell}]$. Let $\ell \leq k$ and $(t_n)_{n=0}^{\infty}$ be an ℓ-admissible point sequence. Then, this sequence is called ℓ-regular with parameter $\gamma \geq 1$ if

$$\frac{|D_{n,i}^{(\ell)}|}{\gamma} \leq |D_{n,i+1}^{(\ell)}| \leq \gamma |D_{n,i}^{(\ell)}|, \qquad n \geq 2,\ -\ell \leq i \leq n-2.$$

The following is due to G. Gevorkyan and A. Kamont [28].

Theorem 0.0.1. ([28]) Let $k \geq 1$ and let (t_n) be a k-admissible sequence of knots in $[0,1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order k. Then, $(f_n^{(k)})$ is a basis in $H^1[0,1]$ if and only if (t_n) is k-regular with some parameter $\gamma \geq 1$.

It is easy to see that $(k-1)$-regularity implies k-regularity. Thus, imposing this stronger condition on the knot sequence we get the unconditional basis property of orthonormal spline systems. The following theorem was developed separately by two groups G. Gevorkyan, K. Keryan and M. Passenbrunner, A. Kamont.

Theorem 0.0.2. ([29]) Let (t_n) be a k-admissible sequence of points. Then, the corresponding orthonormal spline system $(f_n^{(k)})$ is an unconditional basis in $H^1[0,1]$ if and only if (t_n) satisfies the $(k-1)$-regularity condition with some parameter $\gamma \geq 1$.

The only result on periodic orthonormal spline systems in $H^1(\mathbb{T})$ is given in [49]. The authors considered the special case $k = 2$, i.e., Franklin system with arbitrary knots. They gave a special geometric characterization of the knots for which the corresponding Franklin system has basis properties. Before stating the result, a few definitions are in order. Let $k \geq 2$ be an integer. Let $(s_n)_{n=1}^{\infty}$ be a k admissible point sequence on the torus $\mathbb{T}$, i.e., a dense sequence of points on the torus $\mathbb{T}$ such that each point occurs at most k times.

For $n \geq k$, we define $\hat{\mathcal{S}}_n$ to be the space of polynomial splines of order k with grid points $(s_j)_{j=1}^{n} \in \mathbb{T}$. For each $n \geq k+1$, the space $\hat{\mathcal{S}}_{n-1}$ has codimension 1 in $\hat{\mathcal{S}}_n$ and, therefore, there exists a function $\hat{f}_n^{(k)} = \hat{f}_n \in \hat{\mathcal{S}}_n$ with $\|\hat{f}_n\|_2 = 1$ that is orthogonal to the space $\hat{\mathcal{S}}_{n-1}$. See that this function $\hat{f}_n$ is unique up to sign. In addition, let $(\hat{f}_n)_{n=1}^{k}$ be an orthonormal basis for $\hat{\mathcal{S}}_k$. The system of functions $(\hat{f}_n^{(k)})_{n=1}^{\infty}$ is called periodic orthonormal spline system of order k corresponding to the sequence $(s_n)_{n=1}^{\infty}$.

Now we define the atomic Hardy space on $\mathbb{T}$. $a : \mathbb{T} \to \mathbb{R}$ is called a periodic atom, if either $a \equiv 1$ or there exists a periodic interval $\Gamma \subset \mathbb{T}$ such that the following conditions are satisfied:

(i) $\operatorname{supp} a \subset \Gamma$,

(ii) $\|a\|_{L^\infty(\mathbb{T})} \leq |\Gamma|^{-1}$,

(iii) $\int_{\mathbb{T}} a(x)\,\mathrm{d}x = \int_{\Gamma} a(x)\,\mathrm{d}x = 0$.

Now, $H^1(\mathbb{T})$ is the family of all those f functions that has representation

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some periodic atoms $(a_n)_{n=1}^{\infty}$ and real scalars $(c_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$. The space $H^1(\mathbb{T})$ becomes a Banach space under the norm

$$\|f\|_{H^1(\mathbb{T})} := \inf \sum_{n=1}^{\infty} |c_n|,$$

where inf is taken over all (periodic) atomic representations $\sum c_n a_n$ of f. Now, we introduce regularity conditions on the torus $\mathbb{T}$ for sequence $(s_n)_{n=1}^{\infty}$.

Assume that $n \geq k+1$. Let $(\sigma_j)_{j=0}^{n-1}$ be the ordered sequence of knot points consisting of $(s_j)_{j=1}^{n}$ on $\mathbb{T}$ canonically identified with $[0,1)$:

$$\hat{\mathcal{T}}_n = (0 \leq \sigma_{n,0} \leq \sigma_{n,1} \leq \cdots \leq \sigma_{n,n-2} \leq \sigma_{n,n-1} < 1).$$

For integer $\ell \le k$ and $i \in \mathbb{N}_0$, we define $T_{n,i}^{(\ell)} := [\sigma_{n,i}, \sigma_{n,i+\ell}] \subset \mathbb{T}$ interval. Here we observe index i periodically, i.e., we use the notation of periodic extension of the sequence $(\sigma_j)_{j=0}^{n-1}$, i.e., $\sigma_{rn+j} = r + \sigma_j$ for $j \in \{0, \ldots, n-1\}$ and $r \in \mathbb{Z}$ and in the subindices of the B-spline functions, we take the indices modulo n.

Let $\ell \le k$ and $(s_n)_{n=1}^{\infty}$ be an ℓ-admissible point sequence on the torus $\mathbb{T}$. Then, this sequence is called ℓ-regular on the torus $\mathbb{T}$ with parameter $\gamma \ge 1$ if

$$\frac{|T_{n,i}^{(\ell)}|}{\gamma} \le |T_{n,i+1}^{(\ell)}| \le \gamma |T_{n,i}^{(\ell)}|, \qquad n \ge \ell+1,\ i \in \mathbb{N}_0.$$

Theorem 0.0.3. ([49]) Let (s_n) be a 2-admissible sequence of knots on $\mathbb{T}$ with the corresponding periodic Franklin system $(\hat{f}_n^{(2)})_{n\ge1}$. Then, $(\hat{f}_n^{(2)})_{n\ge1}$ is a basis in $H^1(\mathbb{T})$ if and only if (s_n) is 2-regular on the torus with some parameter $\gamma \ge 1$.

Theorem 0.0.4. ([49]) Let (s_n) be a 2-admissible sequence of knots on $\mathbb{T}$ with the corresponding periodic Franklin system $(\hat{f}_n^{(2)})_{n\ge1}$. Then, $(\hat{f}_n^{(2)})_{n\ge1}$ is an unconditional basis in $H^1(\mathbb{T})$ if and only if (s_n) is 1-regular on the torus with some parameter $\gamma \ge 1$.

The paper [39] was of considerable importance to our work, offering a thorough analysis of periodic orthonormal spline functions. Several key properties of these functions were pivotal in deriving our main results.

Theorem 0.0.5. ([39]) Let $k \in \mathbb{N}$ and $(s_n)_{n\ge1}$ be an admissible sequence of knots in $\mathbb{T}$. Then the corresponding periodic orthonormal spline system of order k is an unconditional basis in $L^p(\mathbb{T})$ for every $1 < p < \infty$.

In Chapter 1 we show that k-regularity of knot sequences is a necessary and sufficient condition for corresponding orthonormal spline system to be basis in $H^1(\mathbb{T})$, i.e., we get a generalization of Theorem 0.0.3.

First, we give the main result of Chapter 1, which is proven in [2*].

Theorem 1.2.1. Let $k \ge 1$ and let (s_n) be a k-admissible sequence of knots in $\mathbb{T}$ with the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ of the order k. Then, $(\hat{f}_n^{(k)})$ is a basis in $H^1(\mathbb{T})$ if and only if (s_n) is k-regular on the torus with some parameter $\gamma \ge 1$.

We see that the main theorem in the chapter is proven by establishing two propositions that provide bounds for the $H^1(\mathbb{T})$ norm of the orthogonal projection operators $\hat{P}_n^{(k)}$.

Since the sequence of knots $(s_n)_{n=1}^{\infty}$ is dense on the torus $\mathbb{T}$, the linear span of the functions $\{\hat{f}_n^{(k)}, n \ge 1\}$ is linearly dense in $C(\mathbb{T})$, which implies its linear density in $H^1(\mathbb{T})$. Therefore, $\{\hat{f}_n^{(k)}, n \ge 1\}$ is a basis in $H^1(\mathbb{T})$ if and only if the orthogonal projection operators $\hat{P}_n^{(k)}$ are uniformly bounded in $H^1(\mathbb{T})$, i.e., there is a constant $C = C(\mathcal{T})$, that only depends on the admissible knot sequence $\mathcal{T} = (s_n)_{n=1}^{\infty}$, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} = \|\hat{P}_n^{(k)} : H^1(\mathbb{T}) \to H^1(\mathbb{T})\| \le C(\mathcal{T}). \tag{1}$$

We show that (1) is equivalent to k-regularity of $\mathcal{T}$. This is an immediate consequence of the Propositions 1.2.2. and 1.2.3., which give estimates of the norms $\hat{P}_n^{(k)}$ from below and from above, respectively.

Proposition 1.2.2. Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots on the torus $\mathbb{T}$ of multiplicities at most k. Let

$$M = M_n^{(k)} := \max\left\{ \frac{|T_{n,i}^{(k)}|}{|T_{n,i+1}^{(k)}|}, \frac{|T_{n,i+1}^{(k)}|}{|T_{n,i}^{(k)}|} : \quad 0 \le i \le n-1 \right\}.$$

Then there is a constant $C_k > 0$, depending only on k, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \ge C_k \log M_n^{(k)}.$$

Proposition 1.2.3. Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots on the torus $\mathbb{T}$ of multiplicities at most k. Let γ be such that

$$\frac{|T_{n,i}^{(k)}|}{\gamma} \le |T_{n,i+1}^{(k)}| \le \gamma |T_{n,i}^{(k)}|, \qquad n \ge k+1,\ i \in \mathbb{N}_0.$$

Then there is a constant $C_{k,\gamma} > 0$ depending only on k and γ, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \le C_{k,\gamma}.$$

In order to prove Proposition 1.2.2 we needed the periodic version of the claim used in [28] (cf. page 7, estimate (3.4)).

Proposition 1.4.1. Define $\Phi(x) := \max(0, 1/2 - |x/4|)$ and $\Phi_\epsilon(x) = \frac{1}{\epsilon}\Phi(\frac{1}{\epsilon})$, for $x \in [0,1]$. Then, there is a constant $C > 0$ such that

$$\|f\|_{H^1[0,1]} \ge C\|f^*\|_{L^1[0,1]}, \quad \text{where} \quad f^*(x) = \sup_{\epsilon>0} \Big| \int_0^1 \Phi_\epsilon(x-t) f(t) dt \Big|.$$

Using this proposition we prove a key Lemma used in the proof of Proposition 1.2.2.

Lemma 1.4.2. Define the 1-periodic functions $\hat{\Phi}(x) := \max(0, 1/2 - |x/4|)$ and $\hat{\Phi}_\epsilon(x) = \frac{1}{\epsilon}\hat{\Phi}(\frac{1}{\epsilon})$, for $x \in \mathbb{T}$. Then, for some constant $c > 0$ the following holds,

$$\|f\|_{H^1(\mathbb{T})} \ge c\|f^{**}\|_{L^1(\mathbb{T})}, \quad \text{where} \quad f^{**}(x) = \sup_{\epsilon>0} \Big| \int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) f(t) dt \Big|.$$

In Chapter 2 we show that a stronger condition on knot sequences leads to unconditional basis property of orthonormal spline systems - a property that is stronger than the basis property discussed in the previous chapter. This is a generalization of the sufficiency part of Theorem 0.0.4.

The main result of the chapter, established in [3*], is as follows.

Theorem 2.0.1. Let (s_n) be a k-admissible sequence of points on the torus $\mathbb{T}$. If (s_n) satisfies the $(k-1)$-regularity condition on the torus $\mathbb{T}$ with some parameter $\gamma \ge 1$, then the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ is an unconditional basis in $H^1(\mathbb{T})$.

The proof of the result above requires using Theorem 1.2.1 and three equivalent conditions which we present down below.

Let $N(k)$ be a positive integer only depending on spline order k (this number is specified in the proof of one of the propositions) and let $(a_n)_{n \ge N(k)}$ be a sequence of coefficients , define

$$S := \Big(\sum_{n=N(k)}^{\infty} a_n^2 \hat{f}_n^2 \Big)^{1/2} \qquad \text{and} \qquad M := \sup_m \Big| \sum_{n=N(k)}^{m} a_n \hat{f}_n \Big|.$$

If $f \in L^1(\mathbb{T})$, we denote by Sf and Mf the functions S and M corresponding to the coefficient sequence $a_n = \langle f, \hat{f}_n \rangle$, respectively. Consider the following conditions:

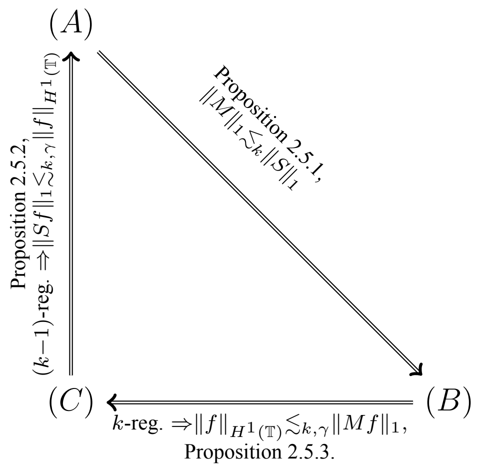
(A) $S \in L^1(\mathbb{T})$,

(B) $M \in L^1(\mathbb{T})$,

(C) There exists a function $f \in H^1(\mathbb{T})$ such that $a_n = \langle f, \hat{f}_n \rangle$.

We prove that under certain regularity conditions on the knot sequence $(s_n)_{n=1}^{\infty}$ the conditions (A)-(C) are equivalent.

Now we discuss the relations between these three conditions. The following diagram illustrates the regularity conditions that must be imposed on the knot sequence $(s_n)_{n=1}^{\infty}$ for the implications to hold.



Let us recall that in case of non-periodic orthonormal spline systems with dyadic knots, the relations (and equivalences) of these conditions have been studied by several authors, see e.g. [52, 10, 18]. For general Franklin systems corresponding to arbitrary sequences of knots, the relations of these conditions were discussed in [27] (and earlier in [25], also in H^p spaces, for $p < 1$, but for a restricted class of point sequences). We follow the approach in [29] and adapt it to the case of periodic orthonormal spline systems of the order k.

The following propositions are used to prove the implications in the diagram above.

Proposition 2.5.1. ($(A) \Rightarrow (B)$) Let (s_n) be a k-admissible sequence of knots on the torus $\mathbb{T}$ and let (a_n) be a sequence of coefficients such that $S \in L^1(\mathbb{T})$. Then, $M \in L^1(\mathbb{T})$ and

$$\|M\|_{L^1(\mathbb{T})} \lesssim_k \|S\|_{L^1(\mathbb{T})}.$$

Proposition 2.5.2. ($(C) \Rightarrow (A)$) Let (s_n) be a k-admissible point sequence on the torus $\mathbb{T}$ that satisfies the $(k-1)$-regularity condition on the torus with parameter $\gamma \geq 1$. Then there exists a constant $C_{k,\gamma}$ depending only on k and γ such that for each atom ϕ,

$$\|S\phi\|_{L^1(\mathbb{T})} \leq C_{k,\gamma}.$$

Consequently, if $f \in H^1(\mathbb{T})$, then

$$\|Sf\|_{L^1(\mathbb{T})} \leq C_{k,\gamma}\|f\|_{H^1(\mathbb{T})}.$$

Proposition 2.5.3. ($(B) \Rightarrow (C)$) Let (s_n) be a k-admissible sequence of knots on the torus $\mathbb{T}$ satisfying the k-regularity condition with parameter $\gamma \geq 1$ and let $(a_n)_{n \geq N(k)}$ be a sequence of coefficients such that $M \in L^1(\mathbb{T})$. Then, there exists a function $f \in H^1(\mathbb{T})$ with $a_n = \langle f, \hat{f}_n \rangle$ for each $n \geq N(k)$. Moreover, we have the inequality

$$\|f\|_{H^1(\mathbb{T})} \lesssim_{k,\gamma} \|Mf\|_{L^1(\mathbb{T})}.$$

In Chapter 3 we see that $(k-1)$-regularity is a necessary condition for $(\hat{f}_n^{(k)})_{n=1}^{\infty}$ to be an unconditional basis in $H^1(\mathbb{T})$. The main result of this chapter, which is proved in [4*], is the following.

Theorem 3.0.1. Let (s_n) be a k-admissible sequence of points on the torus $\mathbb{T}$. If the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ is an unconditional basis in $H^1(\mathbb{T})$, then (s_n) satisfies the $(k-1)$-regularity condition on the torus $\mathbb{T}$ with some parameter $\gamma \geq 1$.

On the other hand, the main result of Chapter 2 gives a sufficient condition for $(\hat{f}_n^{(k)})_{n=1}^{\infty}$ to be an unconditional basis in $H^1(\mathbb{T})$. Thus, combining these two theorems we get the following corollary.

Corollary 3.0.2. Let (s_n) be a k-admissible sequence of points on the torus $\mathbb{T}$. Then, the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ is an unconditional basis in $H^1(\mathbb{T})$ if and only if (s_n) satisfies the $(k-1)$-regularity condition on the torus $\mathbb{T}$ with some parameter $\gamma \geq 1$.

See that Corollary 3.0.2 is a generalization of Theorem 0.0.4.

For the proof of Theorem 3.0.1, we needed the following propositions.

Let $(s_n)_{n=1}^{\infty}$ be a k-admissible sequence of knots on the torus $\mathbb{T}$ with the corresponding periodic orthonormal spline system $(\hat{f}_n)_{n\geq 1}$. For a sequence of coefficients $(a_n)_{n\geq 1}$, let

$$S := \Big(\sum_{n=1}^{\infty} a_n^2 \hat{f}_n^2\Big)^{1/2}.$$

If $f \in L^1(\mathbb{T})$, we denote by Sf the function S corresponding to the coefficient sequence $a_n = \langle f, \hat{f}_n\rangle$. The following proposition is a consequence of Khinchin's inequality.

Proposition 3.3.1. Let (s_n) be a k-admissible sequence of knots on the torus $\mathbb{T}$ with the corresponding periodic orthonormal spline system $(\hat{f}_n)$ and let (a_n) be a sequence of coefficients. If the series $\sum_{n=1}^{\infty} a_n \hat{f}_n$ converges unconditionally in $L^1(\mathbb{T})$, then $S \in L^1(\mathbb{T})$. Moreover,

$$\|S\|_{L^1(\mathbb{T})} \lesssim \sup_{\varepsilon\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{n=1}^{\infty} \varepsilon_n a_n \hat{f}_n \Big\|_{L^1(\mathbb{T})}.$$

Proposition 3.3.2. Let (s_n) be a k-admissible sequence of knots satisfying the k-regularity condition with parameter $\gamma \geq 1$ on the torus $\mathbb{T}$, yet it does not satisfy any $(k-1)$-regularity condition. Then

$$\sup \| \sup_n |a_n(\phi)\hat{f}_n| \|_{L^1(\mathbb{T})} = \infty,$$

where sup is taken over all periodic atoms ϕ and $a_n(\phi) := \langle \phi, \hat{f}_n\rangle$.

For the proof of Proposition 3.2.2 we need the following technical lemma.

Lemma 3.3.3. Let (s_n) be a k-admissible sequence of knots that satisfies the k-regularity condition on the torus $\mathbb{T}$ with parameter $\gamma \geq 1$, but does not satisfy any $(k-1)$-regularity condition. Let ℓ be an arbitrary positive integer. Then, for all $A \geq 2$, there exists a finite increasing sequence $(n_j)_{j=0}^{\ell-1}$ of length ℓ such that if σ_{n_j,i_j} is the new point in $\hat{\mathcal{T}}_{n_j}$ that is not present in $\hat{\mathcal{T}}_{n_j-1}$ and

$$\Lambda_j := [\sigma_{n_j,i_j-k}, \sigma_{n_j,i_j-1}), \qquad L_j := [\sigma_{n_j,i_j-1}, \sigma_{n_j,i_j}), \qquad R_j := [\sigma_{n_j,i_j}, \sigma_{n_j,i_j+1}),$$

we have for all indices i, j in the range $0 \leq i < j \leq \ell - 1$

1. $R_i \cap R_j = \emptyset$,
2. $\Lambda_i = \Lambda_j$,
3. $(2\gamma-1)|L_j| \geq |[\sigma_{n_j,i_j-k-1}, \sigma_{n_j,i_j-k}]| \geq \frac{|L_j|}{2\gamma}$,
4. $|R_j| \leq (2\gamma-1)|L_j|$,

5. $|L_j| \le 2(\gamma+1)k \cdot |R_j|$,

6. $\min(|L_j|,|R_j|) \ge A|\Lambda_j|$.

In Introduction we also give definitions relevant to Chapter 4. Recall the Franklin system. Let $n = 2^\mu + \nu$, where $\mu = 0,1,2,\ldots, 1 \leqslant \nu \leqslant 2^\mu$. Let

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}} & \text{for } 0 \leqslant i \leqslant 2\nu, \\ \frac{i-\nu}{2^\mu} & \text{for } 2\nu < i \leqslant n. \end{cases}$$

Denote by $\mathbf{S}_n$ the space of continuous and piecewise linear functions with nodes $\{s_{n,i}\}_{i=0}^n$, i.e., the function $f \in \mathbf{S}_n$, if $f \in C[0,1]$ and is linear on each interval $[s_{n,i-1}, s_{n,i}]$, $i = 1,2,\ldots,n$. It is easy to see that $\dim \mathbf{S}_n = n+1$ and the set $\{s_{n,i}\}_{i=0}^n$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Therefore, $\mathbf{S}_{n-1} \subset \mathbf{S}_n$ and its codimension is equal to 1. Therefore, there exists a unique function $f_n \in \mathbf{S}_n$ such that f_n is orthogonal to $\mathbf{S}_{n-1}$ in $L^2[0,1]$, $\|f_n\|_2 = 1$ and $f_n(s_{n,2\nu-1}) > 0$. Setting

$$f_0(x) = 1, \quad f_1(x) = \sqrt{3}(2x-1), \quad x \in [0,1],$$

we obtain the orthonormal system $\{f_n(x)\}_{n=0}^\infty$, which was equivalently defined by Franklin in [17].

We say that E is a VP-set for the system $\{\varphi_n\}_{n=0}^\infty$ if from

$$\sum_{n=0}^\infty a_n \varphi_n(x) = f(x), \quad x \notin E,$$

where f is an everywhere finite integrable function, it follows that $\sum_{n=0}^\infty a_n\varphi_n(x)$ is the Fourier series of the function f.

In [22], theorems on VP-sets of the Franklin system were proved.

Theorem 0.0.6. ([22]) The empty set is a VP-set for the Franklin system.

Theorem 0.0.6 follows from Theorem 0.0.7, also proved in [22].

Let

$$\sum_{n=0}^\infty a_n f_n(x) \tag{2}$$

be a Franklin series.

Theorem 0.0.7. ([22]) If the series (2) converges in measure to an integrable function f and the following holds for any x

$$\sup_n \left| \sum_{k=0}^n a_k f_k(x) \right| < \infty,$$

then the series (2) is the Fourier-Franklin series of the function f.

In Chapter 4 we recall some well-known facts about the Franklin system, introducing key definitions, and formulating the main results of the chapter. We also present several lemmas that are used to prove the main results. The main results of this chapter are established in [1*].

Let us denote by $\{\mathbf{f_n}(\mathbf{x})\}_{\mathbf{n}\in\mathbb{N}^k}$ the k-fold Franklin system on $[0,1]^k$, i.e.,

$$\mathbf{f_n}(\mathbf{x}) = f_{n_1}(x_1)\cdots f_{n_k}(x_k), \quad \mathbf{n} \in \mathbb{N}^k, \quad \mathbf{x} \in [0,1]^k,$$

where $\mathbb{N}$ denotes the set of non-negative integers.

We consider multiple Franklin series

$$\sum_{\mathbf{n}\in\mathbb{N}^k} a_{\mathbf{n}}\mathbf{f}_{\mathbf{n}}(\mathbf{x}), \quad \mathbf{x}\in[0,1]^k, \tag{3}$$

with rectangular partial sums

$$S_{\mathrm{N}}(\mathrm{x}) = \sum_{\mathbf{n}\leqslant\mathbf{N}} a_{\mathbf{n}}\mathrm{f}_{\mathbf{n}}(\mathbf{x}), \tag{4}$$

where the notation $\mathbf{n}\leqslant\mathbf{N}$ means that $n_i\leqslant N_i, i=1,\ldots,k$.

A multiple number sequence $S_{\mathbf{N}}$, $\mathbf{N}\in\mathbb{N}^k$, is said to be Pringsheim (or rectangle) convergent to A, written as $\lim_{\mathbf{N}\to\infty} S_{\mathbf{N}} = A$, if

$$\forall\varepsilon>0 \quad \exists M_0\in\mathbb{N}, \quad \text{when} \quad \min_{1\leqslant i\leqslant k} N_i \geqslant M_0, \quad \text{then} \quad |S_{\mathbf{N}}-A|<\varepsilon.$$

Definition 4.1.1. We say that a set $E\subset[0,1]^k$ is sparse in $[0,1]^k$ if

$$E\subset E_0, \quad \text{where} \quad E_0=E_1\times E_2\times\cdots\times E_k, \quad \text{with} \quad \mathrm{mes}\,(E_i)=0, \quad i=1,\ldots,k,$$

where mes(A) is the Lebesgue measure of the set A.

The main results of this chapter are Theorems 4.1.2 and 4.1.3.

Theorem 4.1.2. Let E be a sparse set in $[0,1]^k$ and the series (3) with partial sums (4) satisfies the conditions

$$\sup_{\mathbf{N}}\left|\sum_{\mathbf{n}\leqslant\mathbf{N}} a_{\mathbf{n}}\mathbf{f}_{\mathbf{n}}(\mathrm{x})\right| < \infty, \quad \mathbf{x}\notin E, \tag{5}$$

and there exists the iterated limit

$$\lim_{N_1\to\infty}\cdots\lim_{N_k\to\infty}\sum_{n_1=0}^{N_1} f_{n_1}(x_1)\cdots\sum_{n_k=0}^{N_k} a_{n_1\cdots n_k} f_{n_k}(x_k) =: f(\mathbf{x}), \quad \mathbf{x}\in[0,1]^k, \tag{6}$$

where f is an everywhere finite integrable function. Then the series (3) is the Fourier-Franklin series of the function f.

Theorem 4.1.3. Let E be a sparse set in $[0,1]^k$ and let the series (3) with partial sums (4) satisfy

$$\lim_{\mathbf{N}\to\infty}\sum_{\mathbf{n}\leqslant\mathbf{N}} a_{\mathbf{n}}\mathbf{f}_{\mathbf{n}}(\mathrm{x}) = f(\mathbf{x}), \quad \mathbf{x}\notin E, \tag{7}$$

where f is an everywhere finite integrable function. Then the series (3) is the Fourier-Franklin series of f.

Obviously, any countable set is sparse in $[0,1]^k$. Therefore, Theorem 4.1.3 yields the following.

Theorem 4.1.4. Any countable set is a VP-set for multiple Franklin series converging in the sense of Pringsheim.

References

[1] S. S. Banach, *Théorie des opérations linéaires,* Chelsea, reprint 1955.

[2] N. Bari, *A Treatise on Trigonometric Series,* Pergamon Press, 1964.

[3] S. V. Bochkarev, *Existence of a basis in the space of functions analytic in the disk, and some properties of Franklin's system,* Math. USSR-Sb., **24** (1974), no. 1, 1–16.

[4] S. V. Bochkarev, *Some inequalities for the Franklin series,* Anal. Math., **1**, (1975), 249–257.

[5] W. Böhm, *Inserting new knots into B-spline curves.* Computer-Aided Design, **12** (1980), no. 4, 199–201.

[6] C. de Boor, *on the convergence of odd-degree spline interpolation,* J.Approx. Theory 1 , (1968), 452–463.

[7] G. Cantor, *Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrschen Reihen*, Math. Ann., **5** (1872), 123–132.

[8] L. Carleson, *An explicit unconditional basis in H^1*. Bull. Sci. Math. (2) **104** (1980), no. 4, 405–416.

[9] V. G. Chelidze, Some methods of summation of double series and double integrals, Tbilisi University Press, (in Russian), (1977).

[10] S.-Y. A. Chang and Z. Ciesielski, *Spline characterization of H^1,* Studia Math., **75** (1983), 183–192.

[11] Z. Ciesielski, *Properties of the orthonormal Franklin system,* Studia Math., **23** (1963), 141 – 157.

[12] Z. Ciesielski, *Properties of the orthonormal Franklin system, II,* Studia Math., **27** (1966), 289 – 323.

[13] Z. Ciesielski, *A construction of a basis in $C^1(I^2)$,* Studia Math., **33** (1969), no. 2, 243–247.

[14] Z. Ciesielski, *Orthogonal projections onto spline spaces with arbitrary knots*, in: Function Spaces, CRC Press, (2000), 155–162.

[15] R. A. DeVore and G. G. Lorentz, *Constructive approximation,* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 303, Springer-Verlag, Berlin, 1993.

[16] J. Domsta, *A theorem on B-splines. II. The periodic case.* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **24** (1976), 1077–1084.

[17] Ph. Franklin, *A Set of Continuous Orthogonal Functions.* Math. Ann., (1928), 522–528.

[18] G. G. Gevorkyan, *Some theorems on unconditional convergence and the majorant of Franklin series and their application to ReH^p,* Trudy Mat. Inst. Steklova, (in Russian), **190** (1989), 49–74.

[19] G. G. Gevorkyan, *On series in the Franklin system*, Math. Analysis, (in Russian), **16** (1990), no. 2, 87–114.

[20] G. G. Gevorkyan, *Uniqueness theorems for series in the Franklin system*, Mathematical Notes, **98** (2015), 847–851.

[21] G. G. Gevorkyan, *On the uniqueness of series in the Franklin system*, Sbornik: Mathematics, **207** (2016), no. 12, 30–53.

[22] G. G. Gevorkyan, *Uniqueness theorems for Franklin series converging to integrable functions*, Sbornik: Mathematics, **209** (2018), no. 6, 802–822.

[23] G. G. Gevorkyan, *Uniqueness theorems for Franklin series*, Proc. Steklov Inst. Math. **303** (2018), no. 1, 58–77.

[24] G. G. Gevorkyan, *Uniqueness theorems for one-dimensional and double Franklin series*, Izv. Mat., **84** (2020), no. 5, 3–19.

[25] G. G. Gevorkyan and A. Kamont, *On general Franklin systems,* Dissertationes Math., **374** (1998), 59 pp.

[26] G. G. Gevorkyan and A. Kamont, *Unconditionality of general Franklin systems in* $L^p[0,1], 1 < p < \infty$, Studia Mathematica, **164** (2004), no. 2, 161–204.

[27] G. G. Gevorkyan and A. Kamont, *General Franklin systems as bases in* $H^1[0,1]$, Studia Math., **167** (2005), 259–292.

[28] G. G. Gevorkyan and A. Kamont, *Orthonormal spline systems with arbitrary knots as bases in* $H^1[0,1]$, East Journal on Approximations, **14** (2008), no. 2, 161–182.

[29] G. Gevorkyan, A. Kamont, K. Keryan and M. Passenbrunner, *Unconditionality of orthogonal spline systems in* $H^1[0,1]$, Studia Mathematica, **226** (2015), no. 2, 123–154.

[30] G. G. Gevorkyan and K. A. Navasardyan, *Uniqueness Theorems for Generalized Haar Systems*, Mathematical Notes, **104** (2018), no. 1, 10–24.

[31] G. G. Gevorkyan and K. A. Navasardyan, *Uniqueness Theorems for Series by Vilenkin System*, J. Contemp. Math. Anal., **53** (2018), no.2, 88–99.

[32] L. D. Gogoladze, *Boundedness of convergent mean multiple functional series*, Mat. notes of the Acad of Sci of the USSR, **34** (1983), 917–923.

[33] L. D. Gogoladze, *On the problem of reconstructing the coefficients of convergent multiple function series*, Izv. Mat., **72** (2008), no. 2, 83–90.

[34] F. G. Harutyunyan, *On the uniqueness of series of a Haar system*, Akad. Nauk Armjan. SSR Dokl., (in Russian), **38** (1964), 129–134.

[35] F. G. Harutyunyan and A. A. Talalyan, *Uniqueness of series in Haar and Walsh systems*, Izv. Akad. Nauk SSSR. Ser. Mat., (in Russian), **28** (1964), no. 6, 1391–1408.

[36] S. Karlin, *Total positivity.* Vol. I, Stanford Univ. Press, Stanford, CA, (1968).

[37] B. S. Kashin and A. A. Saakyan, *Orthogonal series,* volume 75 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1989. Translated from the Russian by Ralph P. Boas, Translation edited by Ben Silver.

[38] K. Keryan, *Unconditionality of general periodic spline systems in* $L^p[0,1]$, $1 < p < \infty$., J. Contemp. Math. Anal., **40** (2005), no. 1, 13–55.

[39] K. Keryan and M. Passenbrunner, *Unconditionality of periodic orthonormal spline systems in L^p.*, Studia Mathematica, **248** (2019), no. 1, 57–91.

[40] G. Kozma and A. Olevskii, *Cantor Uniqueness and Multiplicity Along Subsequences*, Algebra and Analysis, **32** (2020), no. 2, 85–106.

[41] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann., **69** (1910), 331–371

[42] B. Maurey, *Isomorphismes entre espaces H^1*. Acta Math., **145** (1980), no. 1-2, 79–120.

[43] M. Passenbrunner, *Unconditionality of orthogonal spline systems in L^p,* Studia Math. **222** (2014), 51–86.

[44] M. Passenbrunner and A. Shadrin, *On almost everywhere convergence of orthogonal spline projections with arbitrary knots*, J. Approx. Theory, **180** (2014), 77–89.

[45] M. Passenbrunner, *Orthogonal Projectors onto spaces of periodic splines*, Journal of Complexity, **42** (2017), 85–93.

[46] M. B. Petrovskaya, On zero-series in the Haar system and uniqueness sets, Izv. Akad. SSR, Ser. Mat., (in Russian), **28** (1964), no. 4, 773–798.

[47] M. G. Plotnikov and Yu. A. Plotnikova, *Decomposition of dyadic measures and unions of closed $\mathscr{U}$-sets for series in a Haar system*, Mat. Sb., **207** (2016), no.3, 137–152.

[48] M. G. Plotnikov, *λ-Convergence of multiple Walsh–Paley series and sets of uniqueness*, Mathematical Notes, **102** (2017), no. 2, 292–301.

[49] M. P. Poghosyan and K. A. Keryan, *General periodic Franklin system as a basis in $H^1[0,1]$*, J. Contemp. Math. Anal., **40** (2005), no. 1, 56–79.

[50] A. Shadrin, *The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture.* Acta Math., **187** (2001), 59–137.

[51] S. Schonefeld, *Schauder bases in spaces of differentiable functions,* Bull. Amer. Math. Soc., **75** (1969), 586–590.

[52] P. Sjölin and J.O. Strömberg, *Basis properties of Hardy spaces,* Ark. Mat., **21** (1983), 111–125.

[53] V. A. Skvorcov, *A theorem of Cantor type for the Haar system*, Vestnik Moskov. Univ. Ser. I Mat. Meh., (in Russian), (1964), no. 5, 3–6.

[54] Sh. T. Tetunashvili, *On some multiple function series and the solution of the uniqueness problem for Pringsheim convergence of multiple trigonometric series*, Mathematics of the USSR-Sbornik, **73** (1992), no. 2, 517–534.

[55] Ch. J. de la Vallee-Poussin, *Sur l'unicit et du d'eveloppement trigonom etrique*, Belg. Bull. Sc., (1912), 702–718.

[56] P. Wojtaszczyk, *The Franklin system is an unconditional basis in H_1,* Arkiv fur Matematik, **20** (1982), no. 1, 293–300.

The author's publications

[1*] G. G. Gevorkyan, L. A. Hakobyan, *Uniqueness Theorems for Multiple Franklin Series Converging over Rectangles*, Mathematical Notes, **109** (2021), 206–217.
[2*] L. Hakobyan and K. Keryan, *Periodic orthonormal spline systems with arbitrary knots as bases in* $H^1(\mathbb{T})$*,* J. Contemp. Math. Anal., **58** (2023), 33–42.
[3*] L. Hakobyan and K. Keryan, *Unconditionality of periodic orthonormal spline systems in* $H^1(\mathbb{T})$*: Sufficiency,* J. Contemp. Math. Anal., **59** (2024), 163–186.
[4*] L. Hakobyan, *Unconditionality of periodic orthonormal spline systems in* $H^1(\mathbb{T})$*: Necessity,* J. Contemp. Math. Anal., **59** (2024), 245–261.

Ամփոփում

Ատենախոսությունը բաղկացած է ներածությունից և չորս գլուխներից։
Առաջին գլուխը նվիրված է Հարդիի ատոմական պարբերական տարածությունում k-րդ կարգի պարբերական օրթոնորմալ սպլայն համակարգերի բազիսի լինելու անհրաժեշտ և բավարար պայմանների ուսումնասիրությանը։ Նշենք, որ օրթոնորմալ սպլայն համակարգերի պարզ օրինակ է ընդհանուր Ֆրանկլինի համակարգը։ Այս համակարգի ֆունկցիաները հանդիսանում են կտոր առ կտոր գծային ֆունկցիաներ, այլ կերպ ասած, երկրորդ կարգի սպլայններ։ Պարբերական օրթոնորմալ սպլայն համակարգը սահմանվում է k թույլատրելի հանգույցների հաջորդականությամբ, որն ամենուրեք խիտ և ամենաշատը k պատիկություն ունեցող հանգույցների հաջորդականություն է։ Քանի որ հանգույցների հաջորդականությունը ամենուրեք խիտ է $\mathbb{T}$ պարբերական միավոր ինտերվալի վրա, ապա $\{\hat{f}_n^{(k)}, n \geq 1\}$ ֆունկցիաների գծային թաղանթը ամենուրեք խիտ է $C(\mathbb{T})$-ում, ինչը իր հերթին ապահովում է ամենուրեք խտություն $H^1(\mathbb{T})$-ում։ Հետևաբար, $\{\hat{f}_n^{(k)}, n \geq 1\}$-ը բազիս է $H^1(\mathbb{T})$-ում այն և միայն այն դեպքում, եթե օրթոգոնալ պրոյեկցիայի օպերատորները $\hat{P}_n^{(k)}$ հավասարաչափ սահմանափակ են $H^1(\mathbb{T})$-ում։ Այնուհետև մենք ցույց ենք տալիս, որ օրթոգոնալ պրոյեկցիայի օպերատորների սահմանափակությունը համարժեք է հանգույցների հաջորդականության k-ռեգուլյար լինելուն $\mathbb{T}$-ի վրա։ Այստեղ մենք k-ռեգուլյարությունը $\mathbb{T}$-ի վրա սահմանում ենք հետևյալ կերպ. դա այն օրթոնորմալ սպլայն համակարգը ծնող հանգույցների հաջորդականություններն են, որոնց համար հարևան պարբերական B-սպլայնների կրիչների երկարությունների հարաբերությունը վերևից հավասարաչափ սահմանափակ են։ Ապացուցելով երկու պնդումներ, որոնցով ցույց ենք տալիս $\mathbb{T}$-ի վրա k-ռեգուլյարության համարժեքությունը օրթոգոնալ պրոյեկցիայի օպերատորների սահմանափակությանը, ստանում ենք մեր հիմնական արդյունքը տվյալ գլխում. եթե (s_n) հանգույցների k-թույլատրելի հաջորդականությունը $\mathbb{T}$ տորի վրա բավարարում է k-ռեգուլյարության պայմանին, ապա համապատասխան k կարգի պարբերական օրթոնորմալ սպլայն համակարգը բազիս է $H^1(\mathbb{T})$ տարածությունում։ Ճիշտ է նաև հակառակը. եթե պարբերական օրթոնորմալ սպլայն համակարգը հանդիսանում է բազիս $H^1(\mathbb{T})$ տարածությունում, ապա համապատասխան հանգույցների հաջորդականությունը բավարարում է k-ռեգուլյարության պայմանին $\mathbb{T}$-ի վրա։
Երկրորդ գլխում դիտարկվում է k կարգի պարբերական օրթոնորմալ սպլայն համակարգերի ոչ պայմանական բազիսության հարցը։ Հեշտ է նկատել որ, $\mathbb{T}$-ի վրա $(k-1)$-ռեգուլյարությունից բխում է k-ռեգուլյարությունը։ Մենք հիմնավորում ենք, որ հանգույցների վրա այս ավելի խիստ պայմանը հանգեցնում է պարբերական օրթոնորմալ սպլայն համակարգերի ոչ պայմանական բազիս լինելու. հատկություն, որն ավելի խիստ է, քան նախորդ գլխում քննարկված դասական բազիսությունը։ Մասնավորապես, ցույց է տրվում, որ $(k-1)$-ռեգուլյարությունը բավարար է համապատասխան պարբերական օրթոնորմալ սպլայն համակարգի՝ $H^1(\mathbb{T})$ տարածությունում, ոչ պայմանական բազիս հանդիսանալու համար։
Ատենախոսության երրորդ գլխում ցույց ենք տալիս, որ եթե պարբերական օրթոնորմալ սպլայն համակարգը հանդիսանում է ոչ պայմանական բազիս $H^1(\mathbb{T})$ տարածությունում, ապա դրան համապատասխանող հանգույցների հաջորդականությունը պետք է լինի $(k-1)$-ռեգուլյար $\mathbb{T}$-ի վրա։ Մյուս կողմից, երկրորդ գլխի հիմնական արդյունքը տալիս է բավարար պայման՝ $(\hat{f}_n^{(k)})_{n=1}^{\infty}$ համակարգի ոչ պայմանական բազիս լինելու համար $H^1(\mathbb{T})$-ում։ Այսպիսով, այս երկու թեորեմների համակցությամբ մենք ստանում ենք, որ պարբերական օրթոնորմալ սպլայն համակարգը բազիս է $H^1(\mathbb{T})$-ում այն և միայն այն դեպքում, եթե համապատասխան հանգույցների հաջորդականությունը $(k-1)$-ռեգուլյար

է 𝕋-ի վրա:
Չորրորդ գլխում հետազոտվում է Ֆրանկլինի բազմակի համակարգի համար միակության բազմությունները՝ դիտարկելով Պրինգսհեյմի իմաստով զուգամիտությունը: Ապացուցվել են հետևյալ երկու թեորեմները. եթե Ֆրանկլինի բազմակի շարքը Պրինգսհեյմի իմաստով զուգամիտում է ամենուրեք վերջավոր ինտեգրելի ֆունկցիայի, բացառությամբ գուցե որոշ տիպի ամենուրեք նոսր բազմության վրա, ապա այդ շարքը հանդիսանում է սահմանային ֆունկցիայի Ֆուրիե-Ֆրանկլինի շարքը: Բացի այդ, եթե տեղի ունի որոշակի տեխնիկական պայման և Ֆրանկլինի բազմակի շարքի մասնակի գումարներն ունեն հաջորդական սահման, որն ամենուրեք վերջավոր ինտեգրելի ֆունկցիա է, ապա այդ շարքը հանդիսանում է սահմանային ֆունկցիայի Ֆուրիե-Ֆրանկլինի շարքը:

Заключение

Диссертация состоит из введения и четырех глав.

Первая глава посвящена изучению необходимых и достаточных условий для того, чтобы периодическая ортонормальная сплайн-система порядка k была базисом в атомном периодическом пространстве Харди. Отметим, что простым примером ортонормальных сплайн-систем является общая система Франклина, функции которой являются кусочно-линейными, то есть сплайнами второго порядка. Периодическая ортонормальная сплайн-система определяется последовательностью k-допустимых узлов, которая является всюду плотной и имеет кратность не более k. Так как последовательность узлов всюду плотна на $\mathbb{T}$, то линейная оболочка функций $\{\hat{f}_n^{(k)}, n \geq 1\}$ является всюду плотной в $C(\mathbb{T})$, что, в свою очередь, обеспечивает всюду плотность в $H^1(\mathbb{T})$. Таким образом, система $\{\hat{f}_n^{(k)}, n \geq 1\}$ является базисом в $H^1(\mathbb{T})$ тогда и только тогда, когда операторы ортогональной проекции $\hat{P}_n^{(k)}$ равномерно ограничены в $H^1(\mathbb{T})$. Используя этот факт, мы доказываем, что ограниченность операторов ортогональной проекции эквивалентна k-регулярности последовательности узлов на $\mathbb{T}$. Здесь мы определяем k-регулярность на $\mathbb{T}$ следующим образом: Это последовательности узлов, образующие ортонормальную сплайн-систему, для которой отношение длин носителей соседних периодических B-сплайнов равномерно ограничено сверху. Доказывая два утверждения, устанавливающие эквивалентность k-регулярности на $\mathbb{T}$ и ограниченности операторов ортогональной проекции, мы приходим к главному результату данной главы: если k-допустимая последовательность узлов (s_n) на $\mathbb{T}$, который является периодическим единичным интервалом, удовлетворяет условию k-регулярности на $\mathbb{T}$, то соответствующая периодическая ортонормальная сплайн-система порядка k является базисом в $H^1(\mathbb{T})$. Верно и обратное: если периодическая ортонормальная сплайн-система является базисом в $H^1(\mathbb{T})$, то соответствующая последовательность узлов удовлетворяет условию k-регулярности на $\mathbb{T}$.

Во второй главе рассматривается вопрос безусловной базисности периодических ортонормальных сплайн-систем порядка k. Легко заметить, что на $\mathbb{T}$ из $(k{-}1)$-регулярности следует k-регулярность. Мы доказываем, что более строгие условия на последовательность узлов приводят к тому, что соответствующая ортонормальная сплайн-система становится безусловным базисом, что является более строгим свойством, чем классическая базисность, рассмотренная в первой главе. В частности, показано, что $(k{-}1)$-регулярность на $\mathbb{T}$ достаточна для того, чтобы соответствующая периодическая ортонормальная сплайн-система была безусловным базисом в $H^1(\mathbb{T})$.

В третьей главе диссертации доказывается, что если периодическая ортонормальная сплайн-система является безусловным базисом в $H^1(\mathbb{T})$, то соответствующая последовательность узлов должна быть $(k{-}1)$-регулярной на $\mathbb{T}$. С другой стороны, основной результат второй главы предоставляет достаточное условие для того, чтобы система $(\hat{f}_n^{(k)})_{n=1}^{\infty}$ была безусловным базисом в $H^1(\mathbb{T})$. Таким образом, объединяя эти две теоремы, мы получаем, что периодическая ортонормальная сплайн-система является базисом в $H^1(\mathbb{T})$ тогда и только тогда, когда соответствующая последовательность узлов $(k{-}1)$-регулярна на $\mathbb{T}$.

В четвертой главе исследуются множества единственности для кратных систем Франклина с точки зрения сходимости в смысле Прингсхейма. Были доказаны следующие две теоремы: Если кратный ряд Франклина сходится в смысле Прингсхейма к всюду конечной интегрируемой функции, за исключением, возможно, некоторого нигде не плотного множества, то этот ряд является рядом Фурье-Франклина предельной функции. Более того, если выполняется определенное техническое условие, и частичная сумма кратного ряда Франклина имеет последовательный предел, являющийся всюду конечной интегрируемой функцией, то этот ряд также является рядом Фурье-Франклина предельной функции.