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Գագիկ Կարենի Վարդանյան

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On the Gasca-Maeztu conjecture

SYNOPSIS

Of dissertation for requesting the degree of candidate of
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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում

Գիտական ղեկավար՝	Ֆիզ. մաթ. գիտ. դոկտոր Հ. Ա. Հակոբյան
Պաշտոնական ընդդիմախոսներ՝	Ֆիզ. մաթ. գիտ. դոկտոր Մ. Գ. Գրիգորյան Ֆիզ. մաթ. գիտ. թեկնածու Դ. Ս. Ոսկանյան
Առաջատար կազմակերպություն՝	ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտ

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The topic of the dissertation is approved at Yerevan State University.

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The defense of the thesis will be held on June 17, 2026 at 16^{00} at the meeting of the Specialized Council 050 “Computational Mathematics” of the Higher Education and Science Committee of the Republic of Armenia at Yerevan State University (1 A. Manukyan St., Yerevan 0025, Armenia).

The dissertation is available in the library of Yerevan State University.

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Actuality of the subject. Polynomial interpolation is one of the fundamental tools in computational mathematics, numerical analysis, and approximation theory. The basic problem—constructing a polynomial that takes prescribed values at prescribed points—arises in many applications, including function approximation, numerical integration, data fitting, and the construction of cubature formulae.

In the univariate case the theory is classical and complete. The foundational works of Lagrange and Newton provide both the theoretical framework and effective computational procedures: any set of $n+1$ distinct points uniquely determines an interpolating polynomial of degree at most n.

The situation changes substantially in two and more variables. Multivariate polynomial interpolation depends not only on the number of nodes but, crucially, on their geometric configuration. This feature leads to a close connection with algebraic geometry: the correctness (unisolvence) of an interpolation problem can be characterized by the absence of nonzero polynomials of a given degree vanishing on the node set, equivalently, by incidence restrictions with algebraic curves.

A central role in this theory is played by the GC_n-sets introduced by Chung and Yao (1977). These are n-correct node sets for which the fundamental polynomial of each node splits completely into linear factors. Such configurations yield explicit interpolation representations of Lagrange type and are therefore of special interest both from the theoretical and the constructive viewpoints.

The Gasca–Maeztu conjecture (1982) asserts that every GC_n-set contains a maximal line, that is, $n+1$ collinear nodes. The conjecture has been verified only for small degrees: the case $n = 3$ was proved by Gasca and Maeztu, the case $n = 4$ was established by Busch (1990), and the case $n = 5$ was resolved by Hakopian, Jetter, and Zimmermann (2014). For $n \geq 6$ the conjecture remains open, and further progress depends on obtaining structural restrictions on GC_n configurations. For a generalization of the GM conjecture to maximal curves see [19].

Purpose and goals of the thesis. The main purpose of this thesis is to investigate the structure of GC_n-sets and to study the Gasca–Maeztu conjecture, with particular focus on the cases $n = 5$ and $n = 6$.

The thesis consists of five chapters. In the first chapter, we present the necessary background on univariate and bivariate polynomial interpolation, introduce n-correct sets and GC_n-sets, and formulate the Gasca–Maeztu conjecture.

In the second chapter, we study n-node lines in GC_n-sets. We introduce the concept of defect and investigate the relationship between the defect and the number of n-node lines passing through a given node. New proofs of the two fundamental properties of n-node lines in GC_n-sets are presented. Bounds on the number of n-node lines passing through a given node in a GC_n-set are obtained.

In the third chapter, we present a new proof of the Gasca–Maeztu conjecture for $n = 5$. So far, the conjecture has been confirmed only for $n \leq 5$. For the $n = 4$ case, many proofs are known; see e.g. [2], [4], [14]. For the case $n = 5$, no proof except [15] was known. Let us mention that our proof is much shorter and transparent.

In the fourth chapter, we take a step towards proving the Gasca–Maeztu conjecture for $n = 6$. It is worth mentioning that the analogue of the main result of this chapter was a crucial step in the proofs of the cases $n = 4$ and $n = 5$.

And the fifth chapter is devoted to the study of maximal curves in n-correct sets. We present new properties of maximal curves, as well as extensions of known properties.

The object of research. Bivariate polynomial interpolation, n-correct set, n-independent set, n-fundamental polynomial, the Gasca–Maeztu conjecture, GC_n-set, maximal line, maximal curve, n-node line.

The methods of research. The methods of univariate and multivariate polynomial interpolations as well as some methods of algebraic geometry are applied.

Scientific novelty. We develop a systematic theory of maximal curves in n-correct sets. We present new properties of maximal curves, as well as extensions of known properties. We establish characterizations involving the Hilbert function and results on the intersection of two and three maximal curves. We investigate conditions under which maximal curves can share a common component.

We introduce and study the concept of the defect of a line with respect to a GC_n-set. For a k-node line ℓ, the defect measures how many nodes fail to use the line ℓ. We establish new results relating the defect to the structure of the node set, and we prove bounds on the number of n-node lines that can pass through a given node.

We provide a new proof of the Gasca–Maeztu conjecture for the case $n = 5$. The original proof [15] (2014) was the only existing proof for this case. Our proof is significantly shorter and more transparent.

We make progress toward proving the Gasca–Maeztu conjecture for the case $n = 6$. Let us mention that the analogue of this result was crucial in the proofs of the cases $n = 4$ and $n = 5$.

Finally, we develop a systematic theory of maximal curves in n-correct sets. We present new properties of maximal curves, as well as extensions of known properties. We establish characterizations involving the Hilbert function and results on the intersection of two and three maximal curves. We investigate a characterization of maximal curves in GC_n sets.

Practical significance. The results obtained in the thesis are theoretical. They have also practical importance.

The following provisions are presented for the defence.

- New proofs of the fundamental properties of n-node lines in GC_n-sets.
- New properties of n-node lines in GC_n-sets.
- A connection between the defect of a GC_n-set and the structure of n-node lines.
- A new proof of the Gasca–Maeztu conjecture for $n = 5$.
- A result for the case $n = 6$ of the Gasca–Maeztu conjecture.
- New properties of maximal curves in n-correct sets.
- Results on the intersection of two and three maximal curves.
- A characterization of maximal curves in GC_n sets is provided.
- General constructions of n-correct sets with two prescribed maximal curves.

The approbation of obtained results. The results of the thesis were reported

- in the scientific seminars held in the department of Numerical Analysis of Faculty of Informatics and Applied Mathematics of Yerevan State University,
- in the International Conference on Mathematical Analysis and Differential Equations, 19–23 September, 2022, Tsaghkadzor, Armenia.

Publications. The results of the thesis were published in 3 scientific articles and reported in an international conference. One more paper has been accepted for publication.

The structure and the content of the thesis. The thesis consists of introduction, two parts each of which contains three chapters, summary and bibliography.

THE CONTENT OF THE THESIS

In **Chapter 1** we present bivariate interpolation and some basic known facts.

Denote by Π_n the space of bivariate polynomials of total degree $\leq n$, for which

$$N := N_n := \dim \Pi_n = \frac{(n+1)(n+2)}{2}.$$

The space of polynomials in two variables is denoted by Π.

Let $\mathcal{X} := \mathcal{X}_s = \{(x_1, y_1), \dots, (x_s, y_s)\}$ be a set of s distinct nodes in the plane.

The problem of finding a polynomial $p \in \Pi_n$ satisfying the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, 2, \dots, s, \tag{1.1.1}$$

for a data $\bar{c} := \{c_1, \dots, c_s\}$ is called interpolation problem.

Definition 1.2.1. A set of nodes $\mathcal{X}_s$ is called n-solvable if for any data $\bar{c}$ there exists a polynomial $p \in \Pi_n$ satisfying the conditions (1.1.1).

Definition 1.2.2. A set of nodes $\mathcal{X}_s$ is called n-correct if for any data $\bar{c}$ there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions (1.1.1).

A necessary condition of n-correctness is: $\#\mathcal{X}_s = s = N$.

Denote by $p|_{\mathcal{X}}$ the restriction of p on $\mathcal{X}$.

Proposition 1.2.3. The n-correctness of the set $\mathcal{X} := \mathcal{X}_N$ is equivalent to each of the following conditions:

(i) The set $\mathcal{X}$ is n-solvable.

(ii) $p \in \Pi_n$, $p|_{\mathcal{X}} = 0 \implies p = 0$.

Definition 1.2.4. A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for $A := (x_k, y_k) \in \mathcal{X}$, if $p|_{\mathcal{X} \setminus \{A\}} = 0$ and $p(A) = 1$.

We denote an n-fundamental polynomial of $A \in \mathcal{X}$ by $p_A^\star = p_{A,\mathcal{X}}^\star$.

Definition 1.2.5. A set of nodes $\mathcal{X}$ is called n-independent if each node has an n-fundamental polynomial. Otherwise, it is n-dependent. A set $\mathcal{X}$ is called essentially n-dependent if none of its nodes has an n-fundamental polynomial.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $\#\mathcal{X}_s = s \leq N$.

If a set of nodes $\mathcal{X}_s$ is n-independent then the following Lagrange formula gives a polynomial $p \in \Pi_n$ satisfying the conditions (1.1.1):

$$p(x, y) = \sum_{i=1}^{s} c_i p_i^\star(x, y).$$

Proposition 1.2.6. A set of nodes $\mathcal{X}$ is n-independent if and only if it is n-solvable.

Proposition 1.2.7. Any n-independent set of nodes $\mathcal{X}_s$ with the cardinality $s < N$ can be enlarged to an n-correct set $\mathcal{X}_N$.

A plane algebraic curve of degree n, $n \geq 1$, is the zero set of some non-zero bivariate polynomial of degree n. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$. In particular, by ℓ we denote a linear polynomial from Π_1 and the line defined by the equation $\ell(x, y) = 0$.

Definition 1.2.8. Let $\mathcal{X}$ be a set of nodes. We say that a line ℓ is a k-node line if it passes through exactly k nodes of $\mathcal{X}$.

Proposition 1.2.9. Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell q$, where $q \in \Pi_{n-1}$.

This implies that at most $n + 1$ nodes of an n-independent set can be collinear. An $(n + 1)$-node line ℓ is called a maximal line.

Denote the set of maximal lines of an n-correct set $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$.

Proposition 1.2.10. Let $\mathcal{X}$ be an n-correct set. Then the following hold:

(i) Any two maximal lines intersect at a node of $\mathcal{X}$.

(ii) Any three maximal lines are not concurrent.

(iii) $\#\mathcal{M}(\mathcal{X}) \leq n + 2$.

Set $d(n, k) := N_n - N_{n-k} = \frac{1}{2}k(2n + 3 - k)$.

Proposition 1.2.11. Let q be an algebraic curve of degree $k \leq n$ with no multiple components. Then the following hold:

(i) Any subset of q containing more than $d(n, k)$ nodes is n-dependent;

(ii) Any subset $\mathcal{X}$ of q containing exactly $d(n, k)$ nodes is n-independent if and only if

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}} = 0 \implies p = qr, \quad r \in \Pi_{n-k}.$$

Thus at most $d(n, k)$ n-independent nodes lie in a curve q of degree $k \leq n$.

Definition 1.2.12. Let $\mathcal{X}$ be an n-correct set. A curve f of degree $k \leq n$ passing through $d(n, k)$ nodes of $\mathcal{X}$ is called a maximal curve.

Since $d(n, n) = N - 1$ we get that each fundamental polynomial of $\mathcal{X}$ is a maximal curve of degree n.

Definition 1.2.14. Let $\mathcal{X}$ be an n-correct set. We say that a node $A \in \mathcal{X}$ uses a line ℓ, if $p_A^* = \ell q$, where $q \in \Pi_{n-1}$.

Lemma 1.2.15. Suppose $\mathcal{X}$ is an n-correct set and a node $A \in \mathcal{X}$ uses a line ℓ. Then ℓ passes through at least two nodes from $\mathcal{X}$, at which q from the above definition does not vanish.

Definition 1.2.16. For a given set of lines $\ell_1, \ldots, \ell_k$, we define $\mathcal{N}_{\ell_1,\ldots,\ell_k}$ to be the set of those nodes in $\mathcal{X}$ which do not lie in any of the lines ℓ_i, and for which at least one of the lines is not used.

In the case of one line ℓ we have

$$\mathcal{N}_\ell = \{A \in \mathcal{X} : A \notin \ell, \text{ and } A \text{ is not using } \ell\}.$$

Proposition 1.2.17. Assume that $\mathcal{X}$ is a GC_n-set, and $\ell_1, \ldots, \ell_k$ are lines. Then the following hold for $\mathcal{N} = \mathcal{N}_{\ell_1,\ldots,\ell_k}$.

(i) If $\mathcal{N}$ is nonempty, then it is essentially $(n-k)$-dependent.

(ii) $\mathcal{N} = \emptyset$ if and only if the product $\ell_1 \cdots \ell_k$ is a maximal curve.

Now we are in a position to present the Gasca–Maeztu conjecture:

Conjecture 1.2.19. For any GC_n-set there exists at least one maximal line.

The GM conjecture is evident for $n = 2$. Till now it has been confirmed for the degrees $n = 3, 4, 5$.

Theorem 1.2.20. Suppose that the Gasca–Maeztu conjecture holds for all degrees up to n. Then any GC_n-set possesses at least three maximal lines.

Corollary 1.2.21. Let $\mathcal{X}$ be a GC_n-set. Suppose that the Gasca–Maeztu conjecture holds for all degrees up to n. Then any node of $\mathcal{X}$ uses a maximal line.

In **Chapter 2** we study n-node lines in GC_n-sets.

We call a node $A \in \mathcal{X}$ type k_m node if exactly k maximal lines of $\mathcal{X}$ pass through A. Thus, according to Proposition 5.1.10, there can be only type $0_m, 1_m$ and 2_m nodes in $\mathcal{X}$.

Proposition 2.1.5 ([5]). Let λ be a maximal line in a GC_n-set $\mathcal{X}$. Then the set $\mathcal{X} \setminus \lambda$ is a GC_{n-1} set.

Definition 2.1.7 ([3]). The defect of an n-correct set $\mathcal{X}$ is the number $\operatorname{def}(\mathcal{X}) := n + 2 - \#M(\mathcal{X})$.

In view of Proposition 5.1.10 we have that $0 \leq \operatorname{def}(\mathcal{X}) \leq n + 2$.

Proposition 2.1.8 ([5]). Let λ be a maximal line of a GC_n-set $\mathcal{X}$. Then we have that

$$\operatorname{def}(\mathcal{X} \setminus \lambda) = \operatorname{def}(\mathcal{X}) \quad \text{or} \quad \operatorname{def}(\mathcal{X}) - 1.$$

Definition 2.1.9. Given an n-correct set $\mathcal{X}$ and a line ℓ, $\mathcal{X}^\ell$ is the subset of nodes of $\mathcal{X}$ which use the line ℓ.

Theorem 2.1.10 ([6]). Let $\mathcal{X}$ be a GC_n-set. Assume that the GM Conjecture holds for all degrees up to n. Then $\operatorname{def}(\mathcal{X}) \in \{0, 1, 2, 3, n-1\}$.

Corollary 2.1.11. Let $\mathcal{X}$ be a GC_n-set. Assume that the GM Conjecture holds for all degrees up to n. Then we have that

(i) there are no 0_m nodes in $\mathcal{X}$ if $\mathrm{def}(\mathcal{X}) \le 1$;

(ii) there is exactly one 0_m node in $\mathcal{X}$ if $\mathrm{def}(\mathcal{X}) = 2$;

(iii) there are exactly three noncollinear 0_m nodes in $\mathcal{X}$ if $\mathrm{def}(\mathcal{X}) = 3$.

Theorem 2.1.12 ([22]). Assume that GM Conjecture holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n-set, $n \ge 4$, and ℓ be an n-node line.

Then one of the following two conditions holds:

1. $|\mathcal{X}^\ell| = \binom{n}{2}$ if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that

$$\mathcal{X}^\ell = \mathcal{X} \setminus (\ell \cup \lambda_0).$$

2. $|\mathcal{X}^\ell| = \binom{n-1}{2}$ if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that

$$\mathcal{X}^\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'').$$

Moreover, if $n = 3$, then the above statement holds with one addition:

3. $|\mathcal{X}^\ell| = 0$ if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect ℓ at three distinct nodes.

Corollary 2.1.13 ([22]). Assume that GM Conjecture holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n-set with exactly three maximal lines, where $n \ge 4$. Then, there are exactly three n-node lines in $\mathcal{X}$, each of which intersects exactly two of the three maximal lines and does not contain a 2_m node.

Denote by $N(\mathcal{X})$ the set of n-node lines in $\mathcal{X}$.

Proposition 2.2.1. Let $\mathcal{X}$ be a GC_3 set, $\mathrm{def}(\mathcal{X}) = 1$ or 2. Then the following hold:

(i) $\#N(\mathcal{X}) \le 4$ or 5 if $\mathrm{def}(\mathcal{X}) = 1$ or 2, respectively;

(ii) Two 3-node lines may not intersect at a node of $\mathcal{X}$ if they both are type $(k, 1, 1)_m$, where $k = 2$ or 1 if $\mathrm{def}(\mathcal{X}) = 1$ or 2, respectively;

(iii) There are no three 3-node lines such that no two of them intersect at a node of $\mathcal{X}$;

(iv) There are no four 3-node lines concurrent at a node of $\mathcal{X}$.

Proposition 2.3.1 ([21]). Let $\mathcal{X}$ be a GC_n-set, where $n \ge 4$. Assume that GM Conjecture holds for all degrees up to n. Then any two n-node lines in $\mathcal{X}$ intersect at a node of $\mathcal{X}$.

Proposition 2.4.1 ([21]). Let $\mathcal{X}$ be a GC_n-set, where $n \ge 4$. Assume that GM Conjecture holds for all degrees up to n. Then we have that $\#N(\mathcal{X}) \le 3$.

Lemma 2.4.2. Let $\mathcal{X}$ be a GC_n-set, where $n \ge 3$. Then we have that four n-node lines cannot be concurrent at a node in $\mathcal{X}$.

Proposition 2.5.1. Let $\mathcal{X}$ be a GC_n-set, where $n \geq 4$ and ℓ_0 be an n-node line with exactly k_0 number of 0_m nodes. Then we have that

$$\operatorname{def}(\mathcal{X}) = k_0 + 1.$$

Moreover, we have that any n-node line in $\mathcal{X}$ contains

(i) not more than one 2_m node;

(ii) exactly s or $s+1$ 1_m nodes, where $s = \#M(\mathcal{X}) - 2 \geq 1$.

Furthermore, if an n-node line contains all types $0_m, 1_m, 2_m$ nodes, then $\operatorname{def}(\mathcal{X}) = 2$.

In **Chapter 3** we present a new, simpler proof of the Gasca–Maeztu conjecture for $n = 5$.

The key result of this chapter is:

Theorem 3.2.1. For any GC_5 set $\mathcal{X}$ of 21 nodes there is a maximal line, i.e., a 6-node line.

To prove the theorem assume by way of contradiction the following.

Assumption. *The set $\mathcal{X}$ is a GC_5 set with no maximal line.*

In view of the conditions $k_1 \geq k_2 \geq \cdots \geq k_n$ and $k_1 + \cdots + k_n = N - 1$, as well as $k_i \geq 2$, the only possible m-d sequences for any node $A \in \mathcal{X}$ are

$$(5,5,5,3,2); \quad (5,5,4,4,2); \quad (5,5,4,3,3); \quad (5,4,4,4,3); \quad (4,4,4,4,4).$$

Proposition 3.2.3 ([15]). Suppose that $\tilde{\ell}$ is a 2-node line. Then $\tilde{\ell}$ can be used by at most one node of $\mathcal{X}$.

Proposition 3.2.4 ([15]). Suppose that $\tilde{\ell}$ is a 3-node line and is used by two nodes $A, B \in \mathcal{X}$. Then there exists a third node C using $\tilde{\ell}$. Furthermore, A, B, and C share three other lines, each passing through five primary nodes. For each of the three nodes, the m-d sequence is $(5,5,5,3,2)$, and the other two nodes are the primary nodes in the respective fifth line. In particular, $\tilde{\ell}$ is used exactly three times.

Proposition 3.2.5 ([15]). Suppose that a line $\tilde{\ell}$ is used by three nodes $A, B, C \in \mathcal{X}$. Then $\tilde{\ell}$ passes through at least three nodes of $\mathcal{X}$.

Corollary 3.2.6 ([15]). Suppose that a line $\tilde{\ell}$ is used by four nodes in $\mathcal{X}$. Then $\tilde{\ell}$ is a 5-node line.

Proposition 3.2.7 ([15]). Suppose that a line $\tilde{\ell}$ is used by five nodes in $\mathcal{X}$. Then $\tilde{\ell}$ is a 5-node line, and it is actually used by exactly six nodes in $\mathcal{X}$. These six nodes form a GC_2 set and share two more lines with five primary nodes each, i.e., each of these six nodes has the m-d sequence $(5,5,5,3,2)$.

Proposition 3.3.1. Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is $(5,5,5,3,2)$.

Lemma 3.3.3 ([15]). (i) The set B is a GC_2 set, and each node $B \in B$ uses the three lines of L_3 and the two lines it uses within B, i.e.,

$$p^*_{B,\mathcal{X}} = \ell_1\ell_2\ell_3 p^*_{B,B}.$$

(ii) No node in A uses any of the lines of L_3.

Definition 3.3.4. We say that a line ℓ is a k_A-node line if it passes through exactly k nodes of A.

Lemma 3.3.5. (i) Assume that a line $\tilde{\ell} \notin L_3$ does not intersect a line $\ell \in L_3$ at a node in $\mathcal{X}$. Then the line $\tilde{\ell}$ can be used at most by one node from A. Moreover, this latter node belongs to $\ell \cap A$.

(ii) If a line ℓ is 0_A or 1_A-node line then no node from A uses the line ℓ.

(iii) If a line ℓ is 2_A-node line then ℓ can be used by at most one node from A.

(iv) Suppose ℓ is a maximal line in B. Then ℓ can be used by at most one node from A.

Proposition 3.3.6. Let $\ell_{B_1M_1}$ be a 5-node line, which is used by all the six nodes of a subset $A_6 \subset A$. Suppose also that ℓ is a 4-node line passing through B_1. If the line ℓ is used by three nodes from A then all these three nodes belong to A_6.

Lemma 3.3.7. Suppose that a line ℓ, passing through B and different from the line ℓ_{BM_1}, is a 3_A-node line. Then ℓ can be used by at most three nodes from A.

Lemma 3.3.8. We have that $m_3(B) \leq 4$.

Lemma 3.4.1 ([15]). Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. By Proposition 3.3.1, for no node of $\mathcal{X}$ the m-d sequence is $(5, 5, 5, 3, 2)$. Then the following hold.

(i) There is no 3-node line and m-node line is used exactly $m-1$ times, where $m = 2, 4, 5$.

(ii) No two lines used by the same node intersect at a node in $\mathcal{X}$.

Corollary 3.4.2. For no node in $\mathcal{X}$ the m-d sequence is $(5, 5, 4, 3, 3)$ or $(5, 4, 4, 4, 3)$.

Proposition 3.4.3. For no node in $\mathcal{X}$ the m-d sequence is $(5, 5, 4, 4, 2)$.

Proposition 3.5.1. For no node in $\mathcal{X}$ the m-d sequence is $(4, 4, 4, 4, 4)$.

In **Chapter 4** we make a step for proving the Gasca–Maeztu conjecture for $n = 6$.

Theorem 4.2.1. Any GC_6 set contains seven collinear nodes.

To make a step for the proof assume by way of contradiction:

Assumption. *The set $\mathcal{X}$ is a GC_6 set without a maximal line.*

The only possible m-distribution sequences for any node $A \in \mathcal{X}$ in the case $n = 6$ with

$N = 28$ are:

$$\begin{array}{llllll} \text{(i)} & (6,6,6,4,3,2) & \text{(ii)} & (6,6,5,5,3,2) & \text{(iii)} & (6,6,5,4,4,2) \\ \text{(iv)} & (6,6,5,4,3,3) & \text{(v)} & (6,6,4,4,4,3) & \text{(vi)} & (6,5,5,5,4,2) \\ \text{(vii)} & (6,5,5,5,3,3) & \text{(viii)} & (6,5,5,4,4,3) & \text{(ix)} & (6,5,4,4,4,4) \\ \text{(x)} & (5,5,5,5,5,2) & \text{(xi)} & (5,5,5,5,4,3) & \text{(xii)} & (5,5,5,4,4,4). \end{array}$$

Theorem 4.1.9 ([20], [16]). Let $i = 1$ or 2. Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-i) + i$ nodes with $1 + i \leq k \leq n - 1$. Then at most $2i$ different curves of degree $\leq k$ pass through all the nodes of $\mathcal{X}$.

Moreover, there are such $2i$ curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but i lie in a maximal curve of degree $k - i$.

Theorem 4.1.10 ([17]). Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-2)+3$ nodes, $3 \leq k \leq n - 1$. Then at most 3 linearly independent curves of degree $\leq k$ pass through all the nodes of $\mathcal{X}$.

Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $k - 1$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $k - 2$.

Theorem 4.1.11 ([18]). A set $\mathcal{X}$ consisting of at most $3n$ nodes is n-dependent if and only if one of the following conditions holds.

1. $n + 2$ nodes are collinear,

2. $2n + 2$ nodes belong to a (possibly reducible) conic,

3. $\#\mathcal{X} = 3n$, and there exist $\gamma \in \Pi_3$ and $\sigma \in \Pi_n$ such that $\mathcal{X} = \gamma \cap \sigma$.

Corollary 4.1.12. A set $\mathcal{X}$ consisting of at most $3n - 1$ nodes is n-dependent if and only if either $n+2$ nodes are collinear, or $2n+2$ nodes belong to a (possibly reducible) conic.

Consider a 2-node line $\widetilde{\ell}$. For the $\widetilde{\ell}$-m-distribution sequence of a node $A \notin \widetilde{\ell}$ there are only the following five possibilities:

$$\begin{array}{llllll} \text{(i)} & (\widetilde{2},6,6,6,4,3) & \text{(ii)} & (\widetilde{2},6,6,5,5,3) & \text{(iii)} & (\widetilde{2},6,6,5,4,4) \\ \text{(vi)} & (\widetilde{2},6,5,5,5,4) & \text{(x)} & (\widetilde{2},5,5,5,5,5). & & \end{array} \tag{4.3.1}$$

Proposition 4.3.1. Assume that $\mathcal{X}$ is a GC_6-set, and suppose that $\widetilde{\ell}$ is a 2-node line. Then $\widetilde{\ell}$ can be used by at most one node $A \in \mathcal{X}$.

Proposition 4.3.2. Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a 3-node line $\widetilde{\ell}$ is used by two nodes $A, B \in \mathcal{X}$. Then there exists a third node C using $\widetilde{\ell}$ and $\widetilde{\ell}$ is used by exactly three nodes of $\mathcal{X}$.

Moreover, A, B, and C share four other lines with either $6,6,6,4$ or $6,6,5,5$ primary nodes, respectively. Furthermore, the m-d sequence of these three nodes is either $(6,6,6,4,\widetilde{3},2)$, or $(6,6,5,5,\widetilde{3},2)$, respectively.

Proposition 4.3.3. Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a 4-node line $\widetilde{\ell}$ is used by three nodes $A,\ B,\ C \in \mathcal{X}$. Then A, B, and C, besides $\widetilde{\ell}$, share four lines with either $6,6,6,3$; $6,6,5,4$; or $6,5,5,5$ primary nodes, respectively.

Proposition 4.3.4. Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that some 4-node line $\widetilde{\ell}$ is used by four nodes $A,\ B,\ C,\ D \in \mathcal{X}$. Then $\widetilde{\ell}$ is used by exactly 6 nodes.

Moreover, besides $\widetilde{\ell}$, these six nodes share also three other lines each passing through 6 primary nodes.

Proposition 4.3.6. Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and $\widetilde{\ell}$ is a 5-node line used by five nodes of $\mathcal{X}$. Then it is used by exactly six nodes.

Moreover, besides $\widetilde{\ell}$, these six nodes share also three other lines passing through $6,6,5$ primary nodes, respectively.

Proposition 4.3.7. Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}$ is a 6-node line. Assume also that $\widetilde{\ell}$ is used by eight nodes of $\mathcal{X}$. Then it is used by exactly ten nodes of $\mathcal{X}$.

Moreover, these ten nodes form a GC_3 set and share two more lines with six primary nodes each. Furthermore, each of these ten nodes has the m-d sequence $(6,6,6,4,3,2)$.

Proposition 4.3.8. Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}_i$, $i = 1,2$, are two disjoint 6-node lines. Assume also that six nodes of $\mathcal{X}$ are using $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$. Then the six nodes besides $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ share either one more line with 6 primary nodes or two more lines each with 5 primary nodes. In the first case the lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are used by exactly ten nodes of $\mathcal{X}$ and in the second case they are used by exactly six nodes of $\mathcal{X}$.

Moreover, in the first and second cases the ten and six nodes form a GC_3 and GC_2 sets, respectively. Furthermore, each of the ten nodes and each of the six nodes has the m-d sequence $(6,6,6,4,3,2)$ and $(6,6,5,5,3,2)$, respectively.

Here is the main result of the chapter:

Proposition 4.3.9. Assume that $\mathcal{X}$ is a GC_6 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is $(6,6,6,4,3,2)$.

Lemma 4.3.10. (i) The set $\mathcal{B}$ is a GC_3 set, and each node $B \in \mathcal{B}$ uses the three lines of $\mathcal{L}_3$ and the three lines it uses within $\mathcal{B}$, i.e.,

$$p^{\star}_{B,\mathcal{X}} = \ell_1\ell_2\ell_3 p^{\star}_{B,\mathcal{B}}.$$

(ii) No node in $\mathcal{A}$ uses any of the lines of $\mathcal{L}_3$.

Lemma 4.3.11. No node in $\mathcal{A}$ has the m-d sequence $(6,6,6,4,3,2)$.

Lemma 4.3.13.

1. Assume that a line $\widetilde{\ell} \notin \mathcal{L}_3$ does not intersect a line $\alpha \in \mathcal{L}_3$ at a node in $\mathcal{X}$. Then the line $\widetilde{\ell}$ can be used by at most 1 node from $\mathcal{A}$. Moreover, this latter node can belong only to α.

2. If ℓ is $0_{\mathcal{A}}$ or $1_{\mathcal{A}}$-node line then no node from $\mathcal{A}$ uses it.

3. If ℓ is $2_{\mathcal{A}}$-node line then it can be used by at most one node from $\mathcal{A}$.

Lemma 4.3.14. Let $\widetilde{\ell}$ be a $3_{\mathcal{A}}$ type line passing through a 2_m-node $B \in \mathcal{B}$. Assume also that the node set $\mathcal{B} \setminus \{\widetilde{\ell}\}$ contains 4 collinear nodes. Then the line $\widetilde{\ell}$ can be used by at most three nodes from $\mathcal{A}$.

Consider all the lines passing through a node $B \in \mathcal{B}$ and at least one more node of $\mathcal{X}$. Denote the set of these lines by $\mathcal{L}(B)$. Let $m_k := m_k(B)$, $k = 1,2,3$, be the number of $k_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$.

Then the following holds:

$$1 \cdot m_1(B) + 2 \cdot m_2(B) + 3 \cdot m_3(B) = \#\mathcal{A} = 18. \tag{4.3.6}$$

Lemma 4.3.15. We have that $m_3(B) \leq 5$.

Lemma 4.3.16. The line ℓ_{01} is a $3_{\mathcal{A}}$ type 5-node line and is used by exactly six nodes from $\mathcal{A}$.

Lemma 4.3.17. The following is true for at least one of $B \in \{B_2, B_3\}$:

Lemma 4.3.18. The set $\mathcal{B}$, except of the lines ℓ_1 and ℓ_2, may have just one more 3-node line, which passes through the nodes B_3, C_0, C_2, provided that the latter nodes are collinear.

Moreover, (ℓ_1, ℓ_2) is the only disjoint pair of 3-node lines in $\mathcal{B}$.

Lemma 4.3.19. There is a type $3_{\mathcal{A}}$ 4-node line through each of the nodes B_0, B_1, and B_2. Moreover, these lines are used by exactly 3 nodes from $\mathcal{A}$.

Lemma 4.3.20. (i) The above three triples are disjoint in this case.

(ii) Suppose the line ℓ_2'' with 6 primary nodes for a triple coincides with one of the lines ℓ_1 or ℓ_2. Then the triple is not a subset of the set $\mathcal{A}_6$.

Lemma 4.3.21. The second 4 in the $\widetilde{\ell}$-m-d sequence (a), in a respective $\widetilde{\ell}$-m line sequence, corresponds to a $3_{\mathcal{A}}$ 4-node line passing through B, B' or B''.

Lemma 4.3.22. Any two triples of nodes corresponding to two distributions of type (a) or (b) are disjoint.

The main result of this chapter is:

Proposition 4.3.9. Assume that $\mathcal{X}$ is a GC_6 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is $(6,6,6,4,3,2)$.

This proposition is analogous to the key step in the proof of the case $n=5$ (see [15], Prop. 3.12; [27], Prop. 2.8), and represents significant progress toward proving the Gasca–Maeztu conjecture for $n=6$.

In **Chapter 5** we study maximal curves in n-correct sets and GC_n-sets.

Set for $n,k \geq 0$,

$$d(n,k) := N_n - N_{n-k}. \tag{5.1.4}$$

Note that for $0 \leq k \leq n+2$ we have:

$$d(n,k) = (n-k+2) + (n-k+3) + \cdots + (n+1) \quad \left[= \frac{k(2n+3-k)}{2} \right]. \tag{5.1.5}$$

While for $k \geq n+1$, in view of the relation (5.1.1), we have:

$$d(n,k) = N_n.$$

Now note that if $0 \leq k \leq \min(m,n)$ then we have that

$$d(n,m) - d(n,k) = d(n-k, m-k). \tag{5.1.6}$$

Indeed, we have that

$$d(n,m) - d(n,k) = N_n - N_{n-m} - (N_n - N_{n-k}) = N_{n-k} - N_{n-m} = d(n-k, m-k).$$

Then note that $m \leq n$ and $0 \leq k \leq n-m+2$ imply

$$d(n,m) - mk = d(n-k, m). \tag{5.1.7}$$

Indeed, we have that

$$d(n,m) - mk = (n-m+2) + (n-m+3) + \cdots + (n+1) - mk$$

$$= (n-m-k+2) + (n-m-k+3) + \cdots + (n-k+1) = d(n-k, m-k).$$

Proposition 5.2.2 ([23]). Let $\mathcal{X}$ be an n-correct set. Then a curve $f \in \Pi$ of degree $k \leq n$ is a maximal curve for $\mathcal{X}$ if and only if it is used by any node of the set $\mathcal{X} \setminus f$.

Corollary 5.2.3 ([23]). Let $\mathcal{X}$ be an n-correct set. Then:

$$f \in \Pi_k \text{ is a maximal curve} \iff \mathcal{X} \setminus f \text{ is an } (n-k)\text{-correct set}.$$

Corollary 5.2.4. Let $\mathcal{X}$ be an n-correct set, f and fh be maximal curves, where $\deg f = k$ and $\deg h = m$. Then the curve h is a maximal curve for the $(n-k)$-correct set $\mathcal{X} \setminus f$.

Corollary 5.2.5. Let $\mathcal{X}$ be an n-correct set, f_1 and f_2 be maximal curves, with no common components. Suppose that $\deg f_i = k_i,\ i = 1, 2$, where $k_1 + k_2 \leq n$. Then:

1. The curve $f_1 f_2$ is a maximal curve of degree $k_1 + k_2$.

2. The curve f_2 is a maximal curve for the $(n - k_1)$-correct set $\mathcal{X} \setminus f_1$.

Remark 5.2.6. If f_1 and f_2 are maximal curves, with no common components, and $\deg f_i = k_i,\ i = 1, 2$, where $k_1 + k_2 \geq n + 1$, then $\mathcal{X} \subset f_1 \cup f_2$.

Denote by $\mathcal{I}(f_1, f_2)$ the set of intersection points of the curves f_1 and f_2. Denote also $\mathcal{I}_{\mathcal{X}}(f_1, f_2) := \mathcal{I}(f_1, f_2) \cap \mathcal{X}$.

Proposition 5.3.1 ([23]). Let $\mathcal{X}$ be an n-correct set, f_1 and f_2 be maximal curves, with no common components. Suppose that $\deg f_i = k_i,\ i = 1, 2$, where $k_1 + k_2 \leq n$. Then the curves f_1 and f_2 intersect at exactly $k_1 k_2$ distinct points, which all are nodes of $\mathcal{X}$:

$$\#\mathcal{I}_{\mathcal{X}}(f_1, f_2) = k_1 k_2.$$

Denote the following *Hilbert function*:

$$H^n_{k,m} := \# R^n_{k,m},$$

where $R^n_{k,m} := R_{k,m} \cap T^{0,0}_n$.

Note that if $k + m \leq n + 2$ then $H^n_{k,m} = km$.

Proposition 5.3.2. Let $\mathcal{X}$ be an n-correct set, f_1 and f_2 be maximal curves, with no common components. Suppose that $\deg f_i = k_i,\ i = 1, 2$. Then the curves f_1 and f_2 intersect at exactly $H^n_{k_1,k_2}$ nodes of $\mathcal{X}$:

$$\#\mathcal{I}_{\mathcal{X}}(f_1, f_2) = H^n_{k_1,k_2}.$$

Proposition 5.3.4. Let $\mathcal{X}$ be an n-correct set, $f_1 = hg_1$ and $f_2 = hg_2$ be any maximal curves, whose greatest common divisor is h. Suppose that $\deg h = m$ and $\deg g_i = s_i,\ i = 1, 2$, where $s := s_1 + s_2 + m \leq n$. Then the following statements hold.

1. The curve $g_1 g_2 h$ is a maximal curve of degree s.

2. The curve h is a maximal curve of degree m.

3. The curves g_1 and g_2 are maximal curves of degrees s_1 and s_2, respectively, for the $(n - m)$-correct set $\mathcal{X} \setminus h$.

4. $\#\mathcal{I}_{\mathcal{X}}(g_1, g_2) = s_1 s_2$ and $h \cap g_1 \cap g_2 \cap \mathcal{X} = \emptyset$.

5. The curves f_1 and f_2 intersect at exactly $d(n,m) + s_1 s_2$ nodes of $\mathcal{X}$:

$$\#\mathcal{I}_{\mathcal{X}}(f_1, f_2) = d(n,m) + s_1 s_2.$$

Proposition 5.3.5. Let $\mathcal{X}$ be an n-correct set, $f_1 = hg_1$ and $f_2 = hg_2$ be any maximal curves, whose greatest common divisor is h. Suppose that $\deg h = m$ and $\deg g_i = s_i,\ i = 1, 2$. Then the curves f_1 and f_2 intersect at exactly

$$\#\mathcal{I}_{\mathcal{X}}(f_1, f_2) = d(n,m) + H^{n-m}_{s_1,s_2}$$

nodes of $\mathcal{X}$.

Proposition 5.4.1 ([23]). Let $\mathcal{X}$ be an n-correct set. Let also f_1, f_2 and f_3 be three maximal curves, such that every two of them have no common components. Suppose also that $\deg f_i = k_i,\ i = 1, 2, 3$, and $k_1 + k_2 + k_3 \leq n + 2$. Then we have that $f_1 \cap f_2 \cap f_3 = \emptyset$.

Proposition 5.4.2. Let the set $\mathcal{X}$ and the curves f_1, f_2 and f_3 be as in Proposition 5.4.1. Let also $k_1 + k_2 + k_3 \geq n + 1$. Then we have the following three expressions for $\#(f_1 \cap f_2 \cap f_3 \cap \mathcal{X})$:

1. $N - d(n, k_1) - d(n, k_2) - d(n, k_3) + H^n_{k_1,k_2} + H^n_{k_2,k_3} + H^n_{k_1,k_3}$,

2. $N - N_{n-k_1} - N_{n-k_2} - N_{n-k_3} + N_{n-k_1-k_2} + N_{n-k_2-k_3} + N_{n-k_1-k_3}$,

3. $N - d(n - k_1, k_2) - d(n - k_2, k_3) - d(n - k_3, k_1)$.

Denote $\sigma := k_1 + k_2 + k_3 - (n + 2)$.

Proposition 5.4.3. Let the set $\mathcal{X}$ and the curves f_1, f_2 and f_3 be as in Proposition 5.4.1. Let also $k_1 + k_2 + k_3 \geq n + 1$, i.e., $\sigma \geq -1$. Then we have the following two expressions for $\#(f_1 \cap f_2 \cap f_3 \cap \mathcal{X})$:

1. $\frac{1}{2}\sigma(\sigma + 1) - N_{k_1+k_2-n-3} - N_{k_2+k_3-n-3} - N_{k_3+k_1-n-3}$,

2. $\frac{1}{2}\sigma(\sigma + 1)$, provided that $k_i + k_j \leq n + 2\ \forall\ 1 \leq i < j \leq 3$.

Proposition 5.5.1. Let $\mathcal{X}$ be GC_n-set and f be a maximal curve of degree $k,\ 1 \leq k \leq n$. Then f is a product of k distinct lines.

Furthermore, if the Gasca-Maeztu conjecture holds for all degrees up to n, then at least one of the above k lines is a maximal line.

Proposition 5.5.2. Let $\mathcal{X}$ be GC_n-set. Suppose that the Gasca-Maeztu conjecture holds for all degrees up to n. Then f is a maximal curve of degree $k,\ 1 \le k \le n$, if and only if

$$f = \ell_1 \cdots \ell_k,$$

where the set $\ell_i \setminus (\ell_1 \cup \cdots \cup \ell_{i-1})$ contains exactly $n+2-i$ nodes of $\mathcal{X},\ i = 1, \ldots, k$.

Definition 5.5.3. A finite set $\mathcal{L}$ of lines is said to be in *general position,* if

1. no two lines of $\mathcal{L}$ are parallel, and

2. no three lines of $\mathcal{L}$ are concurrent.

Let a set $\mathcal{L} = (\ell_1, \ldots, \ell_{n+2})$ of lines be in general position. Then the set $\mathcal{X}$ of $\binom{n+2}{2}$ intersection points of these lines is called *Chung-Yao set* of degree n.

Corollary 5.5.4. Suppose that $\mathcal{X}$ is Chung-Yao lattice of degree n with set of maximal lines $\mathcal{L}$. Suppose also that f is a maximal curve of degree k. Then f is a product of k maximal lines from $\mathcal{L}$.

Furthermore, if f_1 and f_2 are maximal curves of degrees k_1 and k_2, respectively, with no common components, then we have that $k_1 + k_2 \le n + 2$.

Proposition 5.6.1. Let $\delta = 0$ or 1. Then $\mathcal{X} := \mathcal{X}(\delta)$ is an n-correct set, where $n = m + k - 2 + \delta$. Moreover, f and g are maximal curves of degrees m and k, respectively.

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ԵԶՐԱԿԱՑՈՒԹՅՈՒՆ

Աշխատանքը նվիրված է երկչափ GC_n բազմություններին, մաքսիմալ կորերին եւ Գասքա–Մաեզթուի վարկածին (1982):

Հանգույցների X բազմությունը կոչվում է n-ստույգ, եթե նրա համար երկչափ միջարկման խնդիրը միակորեն լուծելի է Π_n -ով՝ $\leq n$ գումարային աստիճանի երկու փոփոխականի բազմանդամներով: Հանգույցների n-ստույգ բազմությունը կանվանենք GC_n բազմություն, եթե յուրաքանչյուր հանգույցի ֆունդամենտալ բազմանդամը հանդիսանում է գծային բազմանդամների արտադրյալ:

Հայտնի է, որ n-ստույգ բազմություններում առավելագույնը $n+1$ հանգույց կարող են պատկանել նույն ուղղին: Ուղիղներն, որոնք անցնում են $n+1$ հանգույցներով կոչվում են մաքսիմալ:

Ըստ Գասքա-Մաեզթուի վարկածի յուրաքանչյուր GC_n բազմության համար գոյություն ունի մաքսիմալ ուղիղ: Մինչ այժմ վարկածը ապացուցված է միայն $n \leq 5$ դեպքերի համար:

Երկրորդ գլխում ուսումնասիրվում են GC_n բազմություններում n-հանգույցանի ուղիղները: Ստացվել է կապ GC_n բազմության դեֆեկտի և n-հանգույցանի ուղիղների միջև՝

$$\mathbf{def}(X) = k_0 + 1,$$

որտեղ k_0-ն n-հանգույցանի ուղիղի վրա գտնվող $0m$-հանգույցների քանակն է: Ստացվել են նաև գնահատականներ տվյալ հանգույցով անցնող n-հանգույցանի ուղիղների քանակի համար: Ապացուցվել է, որ չորս այդպիսի ուղիղներ չեն կարող անցնել մեկ հանգույցով (երբ $n \geq 3$):

Երրորդ գլխում բերվել է Գասքայի–Մաեզթուի վարկածի երկրորդ ապացույց $n = 5$ դեպքի համար,որը էապես ավելի կարճ է և ավելի պարզ քան սկզբնականը (Հակոբյան, Յետտեր, Ցիմերման, 2014): Նշենք, որ ի տարբերություն նախորդ $n \leq 4$ դեպքերի, այս դեպքում մինչ այժմ չկան այլ ապացույցներ:

Չորրորդ գլխում կատարվել է կարևոր քայլ Գասքայի–Մաեզթուի վարկածի $n = 6$ դեպքի ուսումնասիրության ուղղությամբ: Ապացուցվել է, որ եթե GC_6 բազմությունում չկա մաքսիմալ ուղիղ, ապա ոչ մի հանգույց չի կարող ունենալ $(6, 6, 6, 4, 3, 2)$ մաքսիմալ բաշխման հաջորդականություն: Նշենք, որ այս արդյունքի անալոգը վճռորոշ դեր է խաղացել $n = 5$ դեպքի ապացույցում: Բացի այդ, ստացվել են արդյունքներ ≤ 6 հանգույց ունեցող ուղիղների օգտագործման վերաբերյալ, որոնք հնարավորություն են տալիս հերքել հնարավոր բաշխումների մի շարք այլ դեպքեր: Մասնավորապես, $(6, 6, 6, 4, 3, 2)$ բաշխման հերքման արդյունքում 6-հանգույցանի ուղիղների օգտագործումների առավելագույն թիվը նվազեցվել է 10-ից մինչև 7:

Հինգերորդ գլխում ուսումնասիրվում են մաքսիմալ կորերը n-ստույգ բազմություններում: Հայտնի է, որ $k \leq n$ աստիճանի հանրահաշվական կորը կարող է անցնել n-ստույգ բազմության ամենաշատը $d(n, k) := N_n - N_{n-k}$

հանգույցով, որտեղ $N_n = \dim \Pi_n$: Կորերը, որոնք անցնում են ճիշտ $d(n,k)$ հանգույցով կոչվում են մաքսիմալ:

Աշխատանքում ընդհանրացվել են մաքսիմալ կորերի հայտնի հատկություններ ինչպես նաև ապացուցվել են նոր հատկություններ: Մասնավորապես, գտնվել է առանց ընդհանուր կոմպոնենտի երկու մաքսիմալ կորերի հատման կետերի ճշգրիտ թիվը ընդհանուր դեպքում, արտահայտելով այն Հիլբերտի ֆունկցիայի միջոցով՝

$$\#\mathcal{I}_X(f_1, f_2) = H^n_{k_1,k_2} :$$

Գտնվել է նաև m աստիճանի ընդհանուր կոմպոնենտով երկու մաքսիմալ կորերի հատման կետերի ճշգրիտ թիվը, այն է՝

$$\#\mathcal{I}_X(f_1, f_2) = d(n,m) + H^{n-m}_{s_1,s_2}.$$

Այնուհետև ապացուցվել են նոր արդյունքներ երեք մաքսիմալ կորերի հատման վերաբերյալ, որոնցից յուրաքանչյուր երկուսը փոխադարձաբար պարզ են:

Աշխատանքում բնութագրվում են մաքսիմալ կորերը GC_n բազմություններում, ըստ որի՝ k աստիճանի յուրաքանչյուր մաքսիմալ կոր հանդիսանում է k տարբեր ուղիղների արտադրյալ: Ավելին, եթե Գասքա-Մաեզթուի վարկածը ճիշտ է $\leq n$ աստիճանների բազմանդամների համար, ապա այդ k ուղիղներից առնվազն մեկը մաքսիմալ է:

ЗАКЛЮЧЕНИЕ

Работа посвящена исследованию GC_n-множеств на плоскости, максимальных кривых и гипотезы Гаска–Маезту (1982).

Множество узлов называется n-точным, если задача интерполяции единственным образом разрешима в пространстве Π_n многочленов двух переменных суммарной степени не выше n. n-точное множество называется GC_n, если фундаментальный многочлен каждого узла представляется в виде произведения линейных множителей.

Известно, что в n-точном множестве не более чем $n+1$ узлов могут лежать на одной прямой, а прямые, содержащие $(n+1)$ узлов, называются максимальными.

Согласно гипотезе Гаска–Маезту, каждое GC_n-множество содержит максимальную прямую. На сегодняшний день эта гипотеза доказана только для случаев $n \leq 5$.

Во второй главе исследуются n-узловые прямые в GC_n-множествах. Установлена связь между дефектом множества и структурой n-узловых прямых:

$$\mathrm{def}(X) = k_0 + 1,$$

где k_0 — число 0_m-узлов на n-узловой прямой. Получены также оценки числа n-узловых прямых, проходящих через заданный узел. Доказано, что четыре такие прямые не могут проходить через один узел (при $n \geq 3$).

В третьей главе приведено новое доказательство гипотезы Гаска–Маезту для случая $n = 5$, которое существенно короче и проще известного ранее (Акопян, Йеттер, Циммерман, 2014). Отметим, что в отличие от случаев $n \leq 4$, других доказательств для $n = 5$ не известно.

В четвёртой главе сделан важный шаг в исследовании случая $n = 6$. Доказано, что если в GC_6-множестве отсутствует максимальная прямая, то ни один узел не может иметь максимальное распределение $(6, 6, 6, 4, 3, 2)$. Аналогичный результат играл ключевую роль в доказательстве случая $n = 5$. Кроме того, получены результаты об использовании прямых с числом узлов не более 6, что позволяет исключить ряд возможных распределений. В частности, опровержение распределения $(6, 6, 6, 4, 3, 2)$ приводит к уменьшению максимального числа использований 6-узловых прямых с 10 до 7.

В пятой главе исследуются максимальные кривые в n-точных множествах. Известно, что алгебраическая кривая степени $k \leq n$ может проходить не более чем через $d(n,k) := N_n - N_{n-k}$ узлов n-точного множества, а кривые, проходящие ровно через $d(n,k)$ узлов, называются максимальными.

В работе получены новые свойства максимальных кривых и обобщены известные результаты. В частности, найдено точное число точек пересечения двух максимальных кривых без общей компоненты:

$$\#\mathcal{I}_X(f_1, f_2) = H^n_{k_1,k_2}.$$

Также получено точное число точек пересечения двух максимальных кривых с общей компонентой степени m:

$$\#\mathcal{I}_X(f_1, f_2) = d(n, m) + H_{s_1,s_2}^{n-m}.$$

Получены также результаты о пересечении трёх максимальных кривых, каждая пара которых взаимно проста.

Максимальные кривые в GC_n-множествах охарактеризованы: доказано, что любая максимальная кривая степени k представляется в виде произведения k различных прямых. Более того, если гипотеза Гаска–Маезту верна для многочленов степени $\leq n$, то хотя бы одна из этих прямых является максимальной.